A brief introduction to Category Theory
for the working quantum computer scientist

Pablo Andrés-Martínez

November 24, 2022

Category theory takes a bird's eye view of mathematics. From high in the sky, details become invisible, but we can spot patterns that were impossible to detect from ground level.

— Tom Leinster, Basic Category Theory [5]

Preface

This document serves as the lecture notes of a four day course introducing category theory. Its contents have been drawn mainly from the book by Tom Leinster [5] and the book by Chris Heunen and Jamie Vicary [3]. Depending on the reader's background, other books of interest may be Emily Riehl's [6] for the more mathematically inclined and the Dodo book by Bob Coecke and Aleks Kissinger [1] for people particularly interested in graphical calculi for quantum computing.

The course has been designed with the goal of building up the necessary knowledge to discuss two important results regarding categories of quantum processes:

- the universal property of the category of CPTP maps [4] and
- a categorical proof of the Choi-Jamiołkowski isomorphism [7] regarding completely positive maps (rather than CPTP).

Nevertheless, most of the content of the course does not require prior knowledge in quantum computing and may serve as a brief introduction to anyone wanting to dip their toes into category theory.

The exercises in this document are classified according to tags.

- `[!]` General interest. Required to follow the course.
- `[M]` Bonus standard exercise from category theory books.
- `[Q]` Bonus exercise on categories in quantum physics/computing.
- `[C]` Bonus exercise on categories in computer science.

1 Basic definitions

In this first session we will study the three most basic definitions in category theory: category, functor and natural transformation. Examples are provided for each of this concepts. However, before we delve into category theory, let's introduce a generalised notion of group that will be used in many of the examples and exercises.

Definition 1.1. Let A be a set, let $*\colon A \times A \to A$ be a function and let $u \in A$. The triple $(A, *, u)$ is a *monoid* if the following is satisfied:

- $x * u \;=\; x \;=\; u * x$ for all $x \in A$ and
- $(x * y) * z \;=\; x * (y * z)$ for all $x, y, z \in A$.

A *group* is a monoid $(A, *, u)$ that has an inverse element for each $x \in A$ — *i.e.* there is some $y \in A$ such that $x * y = u = y * x$. A *commutative monoid* is a monoid $(A, *, u)$ satisfying $x * y = y * x$ for all $x, y \in A$. An *abelian group* is a group that is also a commutative monoid.

Every mathematical structure has an appropriate notion of structure-preserving map. In this case we have monoid homomorphisms, defined below.

Definition 1.2. Let $(A, *_A, u_A)$ and $(B, *_B, u_B)$ be two monoids. A *monoid homomorphism* is a function $f\colon A \to B$ such that:

- $f(u_A) = u_B$ and
- $f(x *_A y) = f(x) *_B f(y)$ for all $x, y \in A$.

A *group homomorphism* is simply a *monoid homomorphism* between groups.

1.1 Categories

Definition 1.3. A *category* $\mathbf{C}$ is comprised of a collection of *objects* $\mathrm{Ob}(\mathbf{C})$ and, for each pair of objects $A, B \in \mathrm{Ob}(\mathbf{C})$, a collection of *morphisms* $\mathbf{C}(A, B)$, together with a *composition* operation

$$\circ\colon \mathbf{C}(B, C) \times \mathbf{C}(A, B) \to \mathbf{C}(A, C)$$

for all $A, B, C \in \mathrm{Ob}(\mathbf{C})$ satisfying the following axioms.

- *Identities*: For every object A, there is a special morphism $\mathrm{id}_A \in \mathbf{C}(A, A)$ so that
$$f \circ \mathrm{id}_A = f = \mathrm{id}_B \circ f \tag{1}$$
for all $A, B \in \mathrm{Ob}(\mathbf{C})$ and for all $f \in \mathbf{C}(A, B)$,
- *Associativity*: For all $A, B, C, D \in \mathrm{Ob}(\mathbf{C})$ and all $f \in \mathbf{C}(A, B)$, $g \in \mathbf{C}(B, C)$ and $h \in \mathbf{C}(C, D)$,
$$(h \circ g) \circ f = h \circ (g \circ f). \tag{2}$$

For any morphism $f \in \mathbf{C}(A, B)$ we refer to $f\colon A \to B$ as its *type*. To reduce clutter, $A \in \mathbf{C}$ is often used to indicate that A is an object in $\mathrm{Ob}(\mathbf{C})$.

A keen reader may observe that the axioms imposed in the definition of monoid and in the definition of category are essentially the same: associativity and existence of unit/identity element. The only difference is that any pair of elements in a monoid can be multiplied together, but not every pair of morphisms can be composed together — given $f\colon A \to B$ and $g\colon C \to D$ we need that $B = C$ for $g \circ f$ to be well defined. This intuition is captured by the following proposition.

Proposition 1.4. *Let* $\{\bullet\}$ *be an arbitrary singleton set. There is a one-to-one correspondence between monoids* $(A, *, u)$ *and categories* $\mathbf{C}$ *where* $\mathrm{Ob}(\mathbf{C}) = \{\bullet\}$.

Proof. For each monoid $(A, *, u)$ there is a unique category $\mathbf{C}$ where $\mathrm{Ob}(\mathbf{C}) = \{\bullet\}$ and $\mathbf{C}(\bullet, \bullet) = A$ with $\mathrm{id}_\bullet = u$ and $\circ = *$. Associativity and identities in $\mathbf{C}$ follow from the corresponding axioms of monoids. For each category $\mathbf{C}$ where $\mathrm{Ob}(\mathbf{C}) = \{\bullet\}$ it is immediate that $(\mathbf{C}(\bullet, \bullet), \circ, \mathrm{id}_\bullet)$ is a monoid. □

More examples of mathematical concepts that can be realised in terms of categories will be studied in the exercises of this section: in particular, partially ordered sets (Exercise 1.2) and graphs (Exercise 1.5). However, in applied category theory, the most common kind of category is one whose objects are the collection of all mathematical structures of certain kind and whose morphisms are their structure-preserving functions. Some examples of these are provided below.

Example 1.5. The category **Set** has sets as objects and functions as morphisms. More precisely, for any two sets $A, B \in \mathbf{Set}$, the collection $\mathbf{Set}(A, B)$ is the set of all functions from A to B. Composition of functions $(g \circ f)(a) = g(f(a))$ is associative and its identities $\mathrm{id}_A \colon A \to A$ are the usual identity functions.

Example 1.6. The category **Rel** has sets as objects and relations as morphisms. More precisely, for any two sets $A, B \in \mathbf{Rel}$, the collection $\mathbf{Rel}(A, B)$ is the set of all relations between set A and set B, *i.e.* $\mathbf{Rel}(A, B)$ is the powerset of $A \times B$. The composite of $\mathcal{R} \colon A \to B$ and $\mathcal{S} \colon B \to C$ is given as follows:

$$\mathcal{S} \circ \mathcal{R} = \{(a, c) \in A \times C \mid \exists b \in B,\ a\mathcal{R}b \text{ and } b\mathcal{S}c\}.$$

Composition is associative and identities are relations $\mathrm{id}_A = \{(a, a) \mid a \in A\}$.

Example 1.7. The category **Mon** has monoids as objects and each $\mathbf{Mon}(A, B)$ is the set of all monoid homomorphisms from A to B. Composition and identities are the same as in **Set**. Similarly, there is a category **CMon** of commutative monoids, a category **Grp** of groups and a category **Ab** of abelian groups, all of which have monoid homomorphisms as their morphisms.

Example 1.8. The category **Hilb** has Hilbert spaces as objects and each $\mathbf{Hilb}(A, B)$ is the set of all bounded linear maps from A to B. Composition and identities are the same as in **Set**. Similarly, there is a category **FdHilb** whose objects are restricted to finite dimensional Hilbert spaces.

1.2 Functors

Categories are themselves mathematical structures. A natural question arises: is there a notion of structure-preserving maps between categories? Indeed, they are known as functors.

Definition 1.9. Let $\mathbf{C}$ and $\mathbf{D}$ be two categories. A *functor* $F \colon \mathbf{C} \to \mathbf{D}$ is comprised of a mapping between objects so that if $A \in \mathbf{C}$ then $F(A) \in \mathbf{D}$, and a mapping between morphisms so that if $f \in \mathbf{C}(A, B)$ then $F(f) \in \mathbf{D}(F(A), F(B))$. For it to be a functor, F must preserve composition and identities:

$$F(g \circ f) = F(g) \circ F(f)$$
$$F(\mathrm{id}_A) = \mathrm{id}_{F(A)}.$$

Example 1.10. There is a functor **Set** $\to$ **Rel** that sends each set to itself and each function $f\colon A \to B$ to its graph, *i.e.* to the relation:

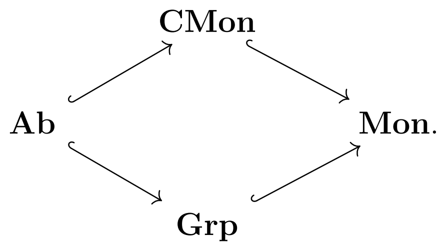
$$\mathcal{R}_f = \{(a,b) \in A \times B \mid f(a) = b\}.$$

The graph of the identity function matches the identity relation; it is easy to check that composition is preserved.

Example 1.11. There is a functor **Mon** $\to$ **Set** that sends each monoid to its underlying set, and each monoid homomorphism to its underlying function. Since composition in **Mon** is defined in the same manner as in **Set**, proving that this is a functor is trivial.

Example 1.12. There is a functor **Ab** $\to$ **CMon** that acts as the identity on objects — since every abelian group is a commutative monoid — and acts as the identity on morphisms. The morphisms and composition in **Ab** are the same as those in **CMon**, so it is trivial to show that this is indeed a functor.

We say that **C** is a *subcategory* of **D** if $\mathrm{Ob}(\mathbf{C})$ is a subcollection of $\mathrm{Ob}(\mathbf{D})$ and $\mathbf{C}(A,B)$ is a subcollection of $\mathbf{D}(A,B)$ for each $A, B \in \mathbf{C}$. If **C** is a subcategory of **D** there is a canonical functor we denote $\mathbf{C} \hookrightarrow \mathbf{D}$.[1] For instance, **Ab** is a subcategory of **CMon** and the functor $\mathbf{Ab} \hookrightarrow \mathbf{CMon}$ was described in the previous example. The following functors exist and describe how certain categories of monoids are subcategories of others:



The examples of functors presented so far have been quite straightforward in that they simply "forget" some information of the objects or morphisms in their source category, *e.g.* **Mon** $\to$ **Set** "forgets" the monoid operation and unit and retrieves the underlying set. In contrast, the following examples provide functors that describe how any set can be upgraded to a monoid so that functions between sets become monoid homomorphisms.

Example 1.13. There is a functor $F\colon \mathbf{Set} \to \mathbf{Mon}$ that maps each set A to the monoid $F(A) = (\texttt{List}(A), +, [\,])$ where $\texttt{List}(A)$ is the collection of finite lists of elements in A, $+$ is list concatenation and $[\,]$ is the empty list. Each function $f\colon A \to B$ is lifted to a monoid homomorphism $F(f)\colon F(A) \to F(B)$ that maps

$$[x, y, \ldots] \mapsto [f(x), f(y), \ldots].$$

It is trivial to check that $F(g \circ f) = F(g) \circ F(f)$ for every two composable functions f and g and $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ for every $A \in \mathbf{Set}$, so F is indeed a functor.

[1] Many authors use $\hookrightarrow$ to refer to functors that act injectively both on the collection of objects and the collection of all morphisms. In this notes we will restrict their use to the case of subcategories.

Example 1.14. There is a functor $G\colon \mathbf{Set} \to \mathbf{CMon}$ that maps each set A to a commutative monoid $G(A) = (\mathtt{Set}(A), \cup, \varnothing)$ where $\mathtt{Set}(A)$ is the collection of finite sets of elements in A, $\cup$ corresponds to union of sets and $\varnothing$ is the empty set. Each function $f\colon A \to B$ is lifted to a monoid homomorphism $G(F)\colon G(A) \to G(B)$ that maps

$$\{x, y, \ldots\} \mapsto \{f(x), f(y), \ldots\}.$$

It is trivial to check that $G(g \circ f) = G(g) \circ G(f)$ for every two composable functions f and g and $G(\mathrm{id}_A) = \mathrm{id}_{G(A)}$ for every $A \in \mathbf{Set}$, so G is indeed a functor.

It is apparent that these two functors $F\colon \mathbf{Set} \to \mathbf{Mon}$ and $G\colon \mathbf{Set} \to \mathbf{CMon}$ are quite similar. The next subsection will provide the tools to "compare" functors and formalise their correspondence.

In the exercise sheet we will see some more examples of functors: the conversion of unitary matrices into completely positive maps is a functor (Exercise 1.3), every graph can be realised as a functor $\mathbf{2} \to \mathbf{Set}$ (Exercise 1.5) and certain programming abstractions such as the `Maybe` monad are functors (Exercise 1.6).

1.3 Natural transformations

So far we have defined categories and maps between categories (functors). Is there a reasonable notion of map between functors? These are known as natural transformations and they are an integral part of category theory.

Definition 1.15. Let $\mathbf{C}$ and $\mathbf{D}$ be categories, let $F\colon \mathbf{C} \to \mathbf{D}$ and $G\colon \mathbf{C} \to \mathbf{D}$ be functors. A *natural transformation* $F \overset{\alpha}{\Rightarrow} G$ is a collection of morphisms $\alpha_A\colon F(A) \to G(A)$ in $\mathbf{D}$ for each $A \in \mathbf{C}$ such that

$$\alpha_B \circ F(f) = HG(f) \circ \alpha_A \tag{3}$$

for every $f \in \mathbf{C}(A, B)$. We say α is a *natural isomorphism* if, additionally, all morphism α_A are invertible.[2]

We will see several examples of natural transformations in the exercises of this section and throughout the course. For now, here is a quick taste of how they look like in practice.

Example 1.16. Take the functors $F\colon \mathbf{Set} \to \mathbf{Mon}$ and $G\colon \mathbf{Set} \to \mathbf{CMon}$ from Examples 1.13 and 1.14. Let $H\colon \mathbf{CMon} \hookrightarrow \mathbf{Mon}$ be the embedding functor acting as the identity on objects and morphisms and let $HG\colon \mathbf{Set} \to \mathbf{Mon}$ be their composition. There is a natural transformation $F \overset{\alpha}{\Rightarrow} HG$ whose component $\alpha_A\colon F(A) \to HG(A)$ for each $A \in \mathbf{Set}$ is the function converting a list into a set, removing duplicated elements.

Proof. To show that indeed $F \overset{\alpha}{\Rightarrow} HG$ is a natural transformation we first need to check that each component α_A is a morphism in $\mathbf{Mon}$, *i.e.* a monoid homomorphism. It is immediate that $\alpha_A([\,]) = \varnothing$ and for any two lists $l_1, l_2 \in F(A)$ it is straightforward to check that $\alpha_A(l_1 + l_2) = \alpha_A(l_1) \cup \alpha_A(l_2)$ since the elements in the resulting sets will be the same no matter on which order we apply these operations. Consequently, α_A for each $A \in \mathbf{Set}$ is indeed a monoid homomorphism. It only remains to show that

$$\alpha_B \circ F(f) = HG(f) \circ \alpha_A$$

[2] A morphism $f\colon A \to B$ is invertible if there is another morphism $g\colon B \to A$ in the category such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$.

for every function $f \in \mathbf{Set}(A, B)$. To check that this holds recall that both $F(f)$ and $G(f)$ simply apply f to each element in the list/set, whereas H acts as identity on morphisms. On one hand, $\alpha_B \circ F(f)$ removes duplicates from the list after applying f to each of its elements; on the other hand, $HG(f) \circ \alpha_A$ removes duplicates first. In either case, though, the result is a set so it cannot have duplicates. Therefore, both sets have the same elements and we conclude that equation (3) holds and $F \overset{\alpha}{\Rightarrow} HG$ is a natural transformation. □

Importantly, the previous example does not simply state that there is a well-defined function that turns every list into a set. The fact that it is a natural transformation means that this conversion turns list concatenation into set union and that it commutes with any function you apply uniformly on the elements of the list/set. Consequently, we have expressed quite a lot of information in just the simple claim that "there is a natural transformation $F \overset{\alpha}{\Rightarrow} HG$". During this course we will see that there is an abundance of examples of categories, functors and natural transformations that are quite distinct from each other at first glance but can be treated in the same abstract terms using category theory.

1.4 Final remarks

A bit on commutative diagrams. In category theory we often use *commutative diagrams* to represent algebraic identities such as the one in the definition of natural transformation (3)

$$\alpha_B \circ F(f) = G(f) \circ \alpha_A.$$

In this particular case, we say that the identity holds if and only if the following diagram commutes:

$$\begin{array}{ccc} F(A) & \xrightarrow{\alpha_A} & G(A) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow{\alpha_B} & G(B). \end{array}$$

A diagram commutes when all directed paths between any two objects compose to the same morphism. These diagrams are nothing more than 2D representations of algebraic identities, but they make reading these identities surprisingly simpler — one of the reasons being that the objects are given explicitly.

Some "meta" categories. Can category theory talk about categories of categories? and categories of functors? Indeed it can, and it is often useful. In this course we will not use these all that much, but some of the exercises in this section will discuss interesting examples of categories of functors.

Definition 1.17. Let **Cat** be the category whose objects are (small)[3] categories and whose morphisms are functors. Composition of functors corresponds to composing their mapping on objects on one hand and their mapping on morphisms on the other. For each category $\mathbf{C}$, the identity functor $1_{\mathbf{C}} \colon \mathbf{C} \to \mathbf{C}$ acts as the identity on objects and morphisms.

[3] A small category is one such that its collection of objects and morphisms are small, *i.e.* they are set. This is to avoid **Cat** (which is not small) from being an object in itself.

Definition 1.18. Let $\mathbf{C}$ and $\mathbf{D}$ be two categories. Let the *functor category* $[\mathbf{C}, \mathbf{D}]$ be the category whose collection of objects is comprised of all functors $\mathbf{C} \to \mathbf{D}$ and whose collection of morphisms $[\mathbf{C}, \mathbf{D}](F, G)$ is comprised of all natural transformations $F \overset{\alpha}{\Rightarrow} G$. Composition of morphisms $F \overset{\alpha}{\Rightarrow} G$ and $G \overset{\beta}{\Rightarrow} H$ corresponds to the natural transformation $F \overset{\beta\circ\alpha}{\Longrightarrow} H$ whose components are $\beta_A \circ \alpha_A$ for each $A \in \mathbf{C}$. For each $F \in [\mathbf{C}, \mathbf{D}]$ the identity morphism id_F is the natural transformation whose components are $\mathrm{id}_{F(A)}$ for each $A \in \mathbf{C}$.

1.5 Exercises

`[!]` **1.1.** Prove that a functor between one-object categories (Proposition 1.4) is a monoid homomorphism.

`[M]` **1.2.** A *partially ordered set* is a pair $(A, \leq)$ where A is a set and $\leq$ is a relation on A satisfying:

- $x \leq x$ for all $x \in A$,
- $x \leq y$ and $y \leq z \implies x \leq z$ for all $x, y, z \in A$,
- $x \leq y$ and $y \leq x \implies x = y$ for all $x, y \in A$.

An *acyclic category* is a category $\mathbf{C}$ where $\mathrm{Ob}(\mathbf{C})$ is a set and for all objects $A, B \in \mathbf{C}$ their set of morphisms $\mathbf{C}(A, B)$ is either singleton or empty and if $A \neq B$ then either $\mathbf{C}(A, B) = \varnothing$ or $\mathbf{C}(B, A) = \varnothing$.

(a) Prove that there is a one-to-one correspondence between partially ordered sets and acyclic categories.

(b) Prove that a functor between acyclic categories is an isotone function.[4]

`[Q]` **1.3.** Let **Isometry** be the category whose objects are finite dimensional Hilbert spaces and whose morphisms are isometries. Let **CPTP** be the category whose objects are finite dimensional Hilbert spaces and whose morphisms $A \to B$ are CPTP maps $\mathcal{L}(A) \to \mathcal{L}(B)$ where $\mathcal{L}(A)$ is the Hilbert space of linear maps of type $A \to A$.

(a) Define a functor **Isometry** $\to$ **CPTP** that sends unitary maps to CPTP maps.

(b) Prove that this is indeed a functor, *i.e.* it preserves composition and identities.

`[!]` **1.4.** Let $F\colon \mathbf{Set} \to \mathbf{Mon}$ be the functor from Example 1.13. Let $G\colon \mathbf{Mon} \to \mathbf{Set}$ be the functor from Example 1.11. Let $FG\colon \mathbf{Mon} \to \mathbf{Mon}$ be their composite. Let $1_{\mathbf{Mon}}\colon \mathbf{Mon} \to \mathbf{Mon}$ be the identity functor.

(a) Let $(A, *, u)$ be a monoid. Let

$$\varepsilon_{(A,*,u)}\colon (\mathtt{List}(A), +, [\,]) \to (A, *, u)$$

be a monoid homomorphism that acts on singleton lists as $[x] \mapsto x$ for all $x \in A$. How does it act on arbitrary elements of $\mathtt{List}(A)$?

[4] Let $(A, \leq)$ and $(B, \preccurlyeq)$ be partially ordered sets; a function $f\colon A \to B$ is *isotone* if $x \leq y \implies f(x) \preccurlyeq f(y)$ for all $x, y \in A$.

(b) Prove that the collection of all such morphisms $\varepsilon_{(A,*,u)}$ defines a natural transformation $FG \overset{\varepsilon}{\Rightarrow} 1_{\mathbf{Mon}}$.

`[M]` **1.5.** Let $\mathbf{2}$ be the category comprised of only two objects and two morphisms (other than the identities) of the following type:

$$\bullet \rightrightarrows \bullet$$

where each $\bullet$ is a distinct object and each arrow is a distinct morphism.

(a) Prove that every graph $G = (V, E)$ is a functor $\mathbf{2} \to \mathbf{Set}$.

(b) A *graph homomorphism* is a function $f \colon V_G \to V_{G'}$ between the set of vertices of two graphs such that if $(u, v) \in E_G$ then $(f(u), f(v)) \in E_{G'}$. Prove that a natural transformation between two graphs seen as functors $\mathbf{2} \to \mathbf{Set}$ is a graph homomorphism.

(c) Provide a functor from the category **Graph** of graphs and graph homomorphisms to the category $[\mathbf{2}, \mathbf{Set}]$ (see Definition 1.18).

`[C]` **1.6.** Let $F \colon \mathbf{Set} \to \mathbf{Set}$ map each set A to $A \uplus \{☹\}$ and each function $f \colon A \to B$ to

$$F(f)(x) = \begin{cases} f(x) & \text{if} \quad x \in A \\ ☹ & \text{otherwise.} \end{cases}$$

(a) Prove that F is a functor.

(b) Define natural transformations $1_{\mathbf{Set}} \overset{\eta}{\Rightarrow} F$ and $FF \overset{\mu}{\Rightarrow} F$ whose components on each $A \in \mathbf{Set}$ map each element $x \in A$ to itself.

(c) Prove that for each $A \in \mathbf{Set}$ the following diagrams commute in **Set**:

$$\begin{array}{ccc} FFF(A) & \xrightarrow{\mu_{F(A)}} & FF(A) \\ {\scriptstyle F(\mu_A)}\downarrow & & \downarrow{\scriptstyle \mu_A} \\ FF(A) & \xrightarrow[\mu_A]{} & F(A) \end{array} \qquad \begin{array}{ccc} F(A) & \xrightarrow{\eta_{F(A)}} & FF(A) \\ {\scriptstyle F(\eta_A)}\downarrow & \searrow^{\mathrm{id}_{F(A)}} & \downarrow{\scriptstyle \mu_A} \\ FF(A) & \xrightarrow[\mu_A]{} & F(A) \end{array}$$

(d) Given two functions $f \colon A \to F(B)$ and $g \colon B \to F(C)$, define a function $g * f \colon A \to F(C)$ that maps each $x \in A$ to $g(f(x)) \in F(C)$. **Hint:** build it using η and μ.

(e) Let $\mathbf{Set}_F$ be a category such that $\mathrm{Ob}(\mathbf{Set}_F) = \mathrm{Ob}(\mathbf{Set})$ and $\mathbf{Set}_F(A, B) = \mathbf{Set}(A, F(B))$ for each $A, B \in \mathbf{Set}$. What are the identity morphisms if we use $*$ as composition?

Note: A functor whose source and target category are the same is known as an *endofunctor*; F is an endofunctor. At the beginning of this document we defined monoids in **Set**, but the same notion may be defined on objects of any category. Hence, we say that (F, μ, η) is a monoid in the category of endofunctors: a *monad*. Indeed, notice that task (c) proves that μ is associative and η is the unit of μ. The category $\mathbf{Set}_F$ is known as the Kleisli category of F; its composition is closely related to `>>=` from Haskell. You may check that F does the same job as the `Maybe` monad in Haskell.

2 Universal properties

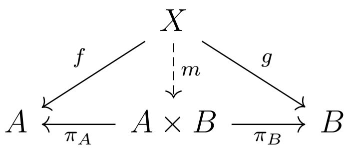
In category theory, whenever we want to talk about a particular object in a category — say, the direct sum $H_0 \oplus H_1$ of two Hilbert spaces — rather than define its contents explicitly, it is best to characterise it in terms of *what makes the object special* in its category. That way, we discover patterns: from a category theory perspective, the empty set plays the same role in **Rel** as the 0-dimensional space in **FdHilb**, whereas the direct sum of Hilbert spaces plays in **FdHilb** a similar role both Cartesian product and disjoint union play in **Set**. This is captured by the concept of *universal property* of which we will see many examples in this section. Certain universal properties appear so often that category theorists have given them names; let's go over some of them.

2.1 Products

Definition 2.1. Let $\mathbf{C}$ be a category and let $A, B \in \mathbf{C}$ be two objects in it. A *product* of A and B is (if it exists):

- an object $A \times B \in \mathbf{C}$ along with
- a morphism $\pi_A \colon A \times B \to A$ and
- a morphism $\pi_B \colon A \times B \to B$

such that for any other object $X \in \mathbf{C}$ and morphisms $f \colon X \to A$ and $g \colon X \to B$ there is a unique morphism $m \colon X \to A \times B$ making the following diagram commute.



Universal properties always involve the *existence* of a *unique* morphism; in this case, for each pair $f \colon X \to A$ and $g \colon X \to B$ there is a unique morphism $m \colon X \to A \times B$ making the diagram commute. The existence of such a morphism m means that, in some loose sense, m subsumes both f and g since we can recover both from it via $f = \pi_A \circ m$ and $g = \pi_B \circ m$. Moreover, since m is required to be unique for each choice of f and g, there is a one-to-one correspondence between pairs of morphisms $(f, g) \in \mathbf{C}(X, A) \times \mathbf{C}(X, B)$ and morphisms $m \in \mathbf{C}(X, A \times B)$ (see Proposition 2.8), *i.e.* for every object $X \in C$ there is a bijection:

$$\mathbf{C}(X, A) \times \mathbf{C}(X, B) \;\cong\; \mathbf{C}(X, A \times B).$$

This is what makes $A \times B$ a special object in $\mathbf{C}$. The examples below show that categorical products are common in mathematics.

Example 2.2. Let $A, B \in \mathbf{Set}$. The Cartesian product of sets A and B

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

along with projections π_A mapping $(a, b) \mapsto a$ and π_B mapping $(a, b) \mapsto b$ comprise a product of A and B in **Set**. For each pair of functions $f \colon X \to A$ and $g \colon X \to B$ the corresponding unique morphism $m \colon X \to A \times B$ is the function

$$m(x) = (f(x), g(x)).$$

Example 2.3. Let $A, B \in \mathbf{Rel}$. The disjoint union of sets A and B

$$A \uplus B = \{a_{\text{left}} \mid a \in A\} \cup \{b_{\text{right}} \mid b \in B\}$$

along with relations $\pi_A = \{(a_{\text{left}}, a) \mid a \in A\}$ and $\pi_B = \{(b_{\text{right}}, b) \mid b \in B\}$ comprise a product of A and B in $\mathbf{Rel}$. For each pair of relations $\mathcal{R}\colon X \to A$ and $\mathcal{S}\colon X \to B$ the corresponding unique morphism $m\colon X \to A \uplus B$ is the relation

$$m = \{(x, a_{\text{left}}) \mid x \in X, a \in A \text{ s.t. } x\mathcal{R}a\} \cup \{(x, b_{\text{right}}) \mid x \in X, b \in B \text{ s.t. } x\mathcal{S}b\}.$$

Example 2.4. Let $A, B \in \mathbf{FdHilb}$. The direct sum of A and B is the vector space $A \oplus B$ on the Cartesian product of their underlying sets, with coordinate-wise addition and scalar multiplication. Then, $A \oplus B$ is made into a Hilbert space by the inner product

$$\langle (a,b) \,|\, (a',b') \rangle = \langle a \,|\, a' \rangle + \langle b \,|\, b' \rangle.$$

The direct sum $A \oplus B$ along with linear maps π_A mapping $(a, b) \mapsto a$ and π_B mapping $(a, b) \mapsto b$ comprise a product of A and B in $\mathbf{FdHilb}$. For each pair of linear maps $f\colon X \to A$ and $g\colon X \to B$ the corresponding unique morphism $m\colon X \to A \oplus B$ is the linear map

$$m(x) = (f(x), g(x)).$$

Example 2.5. Let $(A, \leq)$ be a partially ordered set and view it as a category $\mathbf{C}$ (as in Exercise 1.2). Let $a, b \in A$ and recall that these are objects $a, b \in \text{Ob}(\mathbf{C})$. The category $\mathbf{C}$ has a product $a \times b$ if and only if there is a greatest lower bound for a and b in $(A, \leq)$.[5]

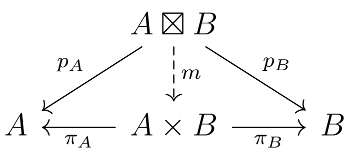
You should convince yourself — by inspecting the commutative diagram from Definition 2.1 — that, indeed, these examples satisfy the definition of product in their respective categories. A non-example is presented below.

Example 2.6. Let $A, B \in \mathbf{Set}$. There are no morphisms $\pi_A\colon A \uplus B \to A$ and $\pi_B\colon A \uplus B \to B$ that make $A \uplus B$ a product in $\mathbf{Set}$. Intuitively, the projections would need to map $\pi_A(a_{\text{left}}) = a$ and $\pi_B(b_{\text{right}}) = b$; however, it is unclear what the value of $\pi_A(b_{\text{right}}) \in A$ should be. For instance, take $A = \{a, a'\}$ and $B = \{b\}$; then, we need to choose whether $\pi_A(b_{\text{right}}) = a$ or $\pi_A(b_{\text{right}}) = a'$. Let's say we choose $\pi_A(b_{\text{right}}) = a$, but this means that π_A provides "two ways to reach a" — either through a_{left} or through b_{right} — implying that the definition of $m\colon X \to A \uplus B$ will not be unique, thus failing the universal property. A formal proof for all $A, B \in \mathbf{Set}$ is sketched after Proposition 2.7.

Perhaps it is not yet convincing that the universal property unambiguously characterises these objects: sure, the direct sum of Hilbert spaces is a categorical product, but if I refer to "the categorical product in $\mathbf{FdHilb}$", am I unequivocally referring to the direct sum? The following two propositions are meant to clarify this point.

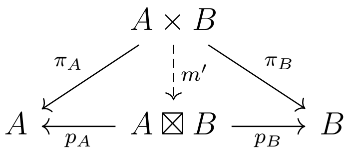
Proposition 2.7. *Let $A, B \in \mathbf{C}$ where both $(A \times B, \pi_A, \pi_B)$ and $(A \boxtimes B, p_A, p_B)$ are products in $\mathbf{C}$. Then, there is an isomorphism $A \times B \cong A \boxtimes B$.*

Proof. Since $A \times B$ is a product, there is a unique morphism m making the following diagram

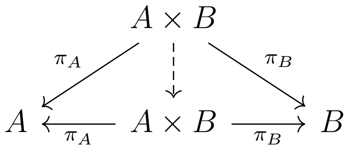


[5]Assuming that $\mathbf{C}$ is defined so that $a \leq b$ implies $\mathbf{C}(a, b) \neq \varnothing$.

commute. Similarly, since $A \boxtimes B$ is a product, there is a unique morphism m' making the following diagram



commute. Finally, since $A \times B$ is a product there is a unique morphism making the following diagram



commute. One such morphism is $\mathrm{id}_{A\times B}$ and it is not difficult to check that another morphism making the diagram commute is $m' \circ m$; however, according to the definition of product, there is only one such morphism (uniqueness), so it must be that $m' \circ m = \mathrm{id}_{A\times B}$. A similar argument can be used to show that $m \circ m' = \mathrm{id}_{A\boxtimes B}$, hence, $m\colon A \times B \to A \boxtimes B$ is an isomorphism whose inverse is m'. □

For instance, you may be familiar with a different definition of the direct sum of two Hilbert spaces $A, B \in \mathbf{FdHilb}$ in terms of a basis given by the disjoint union of a basis for A and a basis for B. Such a definition gives rise to a different Hilbert space, but it is isomorphic to our definition of $A \oplus B$ under appropriate relabelling of the basis vectors. The takeaway is that what matters is the universal property of $A \oplus B$ — *i.e.* that it is a product in **FdHilb** — rather than its explicit construction (of which there may be multiple isomorphic versions).

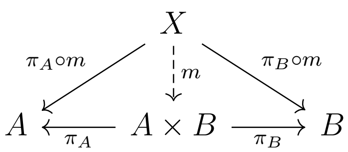
On the other hand, in a previous example we have shown that for each $A, B \in \mathbf{Set}$, their Cartesian product $A \times B$ is a product in **Set**. Since $A \times B$ is *not* isomorphic to $A \uplus B$ — their cardinalities are different — then $A \uplus B$ cannot be a product in **Set**, as claimed in Example 2.6.

Proposition 2.8. *Let* $\mathbf{C}$ *be a category with objects* $A, B \in \mathbf{C}$. *Let* $(A \times B, \pi_A, \pi_B)$ *be a product in* $\mathbf{C}$. *For each* $X \in \mathbf{C}$, *there is a bijection*

$$\mathbf{C}(X, A) \times \mathbf{C}(X, B) \;\cong\; \mathbf{C}(X, A \times B).$$

Proof. Let $(f, g) \in \mathbf{C}(X, A) \times \mathbf{C}(X, B)$; from the definition of product we know that there is a unique morphism $m \in \mathbf{C}(X, A \times B)$ such that $f = \pi_A \circ m$ and $g = \pi_B \circ m$. Let $\phi\colon \mathbf{C}(X, A) \times \mathbf{C}(X, B) \to \mathbf{C}(X, A \times B)$ map each (f, g) to its corresponding m; we need to show that ϕ is injective and surjective.

- Notice that for every $m \in \mathbf{C}(X, A \times B)$ the pair $(\pi_A \circ m, \pi_B \circ m) \in \mathbf{C}(X, A) \times \mathbf{C}(X, B)$ trivially makes the diagram



  commute. Consequently, each $m \in \mathbf{C}(X, A \times B)$ is in the image of ϕ since $\phi(\pi_A \circ m, \pi_B \circ m) = m$, implying ϕ is surjective.

- Let $(f, g) \in \mathbf{C}(X, A) \times \mathbf{C}(X, B)$ and $(f', g') \in \mathbf{C}(X, A) \times \mathbf{C}(X, B)$ satisfy $\phi(f, g) = m = \phi(f', g')$. Then, $\pi_A \circ m = f$ and $\pi_A \circ m = f'$ implying that $f = f'$ and, similarly, $g = g'$. Consequently, ϕ is injective. □

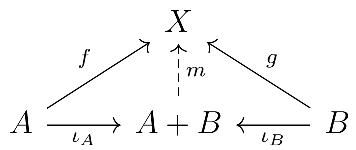
2.2 Coproducts

A common strategy in category theory to get new definitions from previous ones is to *dualise* them. In a hand-wavy way, we say that the dual of a concept is obtained by reversing the direction of all arrows involved in its definition. The dual of a product is a coproduct, explicitly defined below.

Definition 2.9. Let $\mathbf{C}$ be a category and let $A, B \in \mathbf{C}$ be two objects in it. The *coproduct* of A and B is (if it exists):

- an object $A + B \in \mathbf{C}$ along with
- a morphism $\iota_A \colon A \to A + B$ and
- a morphism $\iota_B \colon B \to A + B$

such that for any other object $X \in \mathbf{C}$ and morphisms $f \colon A \to X$ and $g \colon B \to X$ there is a unique morphism $m \colon A+B \to X$ making the following diagram commute.



You may check that Propositions 2.7 and 2.8 hold in the case of coproducts. Namely, given a category $\mathbf{C}$ and objects $A, B \in \mathbf{C}$, any two coproducts of A and B are isomorphic and for any $X \in \mathbf{C}$ there is a bijection

$$\mathbf{C}(A, X) \times \mathbf{C}(B, X) \;\cong\; \mathbf{C}(A + B, X).$$

Some examples of coproducts are given below.

Example 2.10. Let $A, B \in \mathbf{Set}$. The disjoint union of sets A and B

$$A \uplus B = \{a_{\text{left}} \mid a \in A\} \cup \{b_{\text{right}} \mid b \in B\}$$

along with injections ι_A mapping $a \mapsto a_{\text{left}}$ and ι_B mapping $b \mapsto b_{\text{right}}$ comprise a coproduct of A and B in $\mathbf{Set}$. For each pair of functions $f \colon A \to X$ and $g \colon B \to X$ the corresponding unique morphism $m \colon A \uplus B \to X$ is the function

$$m(c) = \begin{cases} f(a) & \text{if} \quad c = a_{\text{left}} \\ g(b) & \text{if} \quad c = b_{\text{right}}. \end{cases}$$

Example 2.11. Let $A, B \in \mathbf{Rel}$. The disjoint union of sets A and B

$$A \uplus B = \{a_{\text{left}} \mid a \in A\} \cup \{b_{\text{right}} \mid b \in B\}$$

along with relations $\iota_A = \{(a, a_{\text{left}}) \mid a \in A\}$ and $\iota_B = \{(b, b_{\text{right}}) \mid b \in B\}$ comprise a coproduct of A and B in $\mathbf{Rel}$. For each pair of relations $\mathcal{R} \colon A \to X$ and $\mathcal{S} \colon B \to X$ the corresponding unique morphism $m \colon A \uplus B \to X$ is the relation

$$m = \{(a_{\text{left}}, x) \mid x \in X, a \in A \text{ s.t. } a\mathcal{R}x\} \cup \{(b_{\text{right}}, x) \mid x \in X, b \in B \text{ s.t. } b\mathcal{S}x\}.$$

Example 2.12. Let $A, B \in \mathbf{FdHilb}$. The direct sum $A \oplus B$ along with linear maps ι_A mapping $a \mapsto (a, 0)$ and ι_B mapping $b \mapsto (0, b)$ comprise a coproduct of A and B in $\mathbf{FdHilb}$. For each pair of linear maps $f \colon A \to X$ and $g \colon B \to X$ the corresponding unique morphism $m \colon A \oplus B \to X$ is the linear map

$$m(a, b) = f(a) + g(b).$$

Example 2.13. Let $(A, \leq)$ be a partially ordered set and view it as a category $\mathbf{C}$ (as in Exercise 1.2). Let $a, b \in A$ and recall that these are objects $a, b \in \mathrm{Ob}(\mathbf{C})$. The category $\mathbf{C}$ has a coproduct $a + b$ if and only if there is a lowest upper bound for a and b in $(A, \leq)$.[6]

2.3 Terminal and initial objects

Definition 2.14. Let $\mathbf{C}$ be a category. A *terminal object* is an object $A \in \mathbf{C}$ such that for each object $X \in \mathbf{C}$ there is a unique morphism of type $X \to A$. Dually, an *initial object* is an object $A \in \mathbf{C}$ such that for each object $X \in \mathbf{C}$ there is a unique morphism of type $A \to X$.

Example 2.15. Let $\{\bullet\}$ be an arbitrary singleton set; $\{\bullet\}$ is a terminal object in $\mathbf{Set}$ and for each set $X \in \mathbf{Set}$ there is a unique function $X \to \{\bullet\}$ mapping each element $x \in X$ to $\bullet$. The empty set $\varnothing$ is an initial object in $\mathbf{Set}$ and for each set $X \in \mathbf{Set}$ there is a unique function $\varnothing \to X$, *i.e.* the trivial function.

Notice that all singleton sets are isomorphic, as expected from objects characterised by the same universal property. Notice that $\varnothing$ cannot be a terminal object in $\mathbf{Set}$ since there is no (total) function $X \to \varnothing$.

Example 2.16. The empty set $\varnothing$ is both an initial and a terminal object in $\mathbf{Rel}$. For each set $X \in \mathbf{Rel}$ there is a unique relation $X \to \varnothing$ and a unique relation $\varnothing \to X$, *i.e.* the empty relation.

Notice that $\{\bullet\}$ is *not* a terminal object in $\mathbf{Rel}$ since for each set X there are $2^{|X|}$ relations $X \to \{\bullet\}$.

Example 2.17. The 0-dimensional Hilbert space $\{0\}$ is both an initial and a terminal object in $\mathbf{FdHilb}$. For each Hilbert space $X \in \mathbf{FdHilb}$ there is a unique linear map $X \to \{0\}$ mapping all vectors to 0 and a unique linear map $\{0\} \to X$ mapping 0 to the zero vector in X.

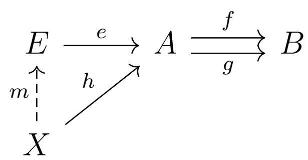
2.4 Equalizers and coequalizers

Following [5], we first introduce a piece of preliminary terminology. Let $\mathbf{C}$ be a category and let $f, g\colon A \to B$ be two morphisms in $\mathbf{C}$. A *fork* of f, g consists of an object $X \in \mathbf{C}$ along with a morphism $h\colon X \to A$ such that the diagram

$$X \xrightarrow{h} A \underset{g}{\overset{f}{\rightrightarrows}} B$$

commutes, *i.e.* such that $f \circ h = g \circ h$.

Definition 2.18. Let $\mathbf{C}$ be a category and let $f, g\colon A \to B$ be two morphisms in $\mathbf{C}$. An *equalizer* of f, g is a fork of f, g — given by E and $e\colon E \to A$ — such that for any other fork of f, g — given by X and $h\colon X \to A$ — there is a unique morphism $m\colon X \to E$ making the following diagram



commute.

[6] Assuming that $\mathbf{C}$ is defined so that $a \leq b$ implies $\mathbf{C}(a, b) \neq \varnothing$.

Example 2.19. Let $f, g\colon A \to B$ be two functions. The equalizer of f, g in **Set** is the set

$$E = \{a \in A \mid f(a) = g(a)\}$$

along with the embedding function $E \hookrightarrow A$.

Example 2.20. Let $f, g \in \mathbf{FdHilb}(A, B)$. The equalizer of f, g in **FdHilb** is the kernel

$$\ker(f - g) = \{v \in A \mid (f - g)(a) = 0\}$$

which is a closed subspace of A and, hence, a Hilbert space; the morphism of the equalizer is the embedding $\ker(f - g) \hookrightarrow A$.

Dually, we can define a *cofork* of $f, g\colon A \to B$ to be an object $X \in \mathbf{C}$ along with a morphism $h\colon B \to X$ such that the diagram

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{\;h\;} X$$

commutes, *i.e.* such that $h \circ f = h \circ g$.

Definition 2.21. Let $\mathbf{C}$ be a category and let $f, g\colon A \to B$ be two morphisms in $\mathbf{C}$. A *coequalizer* of f, g is a cofork of f, g — given by K and $k\colon B \to K$ — such that for any other cofork of f, g — given by X and $h\colon B \to X$ — there is a unique morphism $m\colon K \to X$ making the following diagram

$$\begin{array}{ccccc} A & \underset{g}{\overset{f}{\rightrightarrows}} & B & \xrightarrow{\;k\;} & K \\ & & & \underset{h}{\searrow} & \Big\downarrow{\scriptstyle m} \\ & & & & X \end{array}$$

commute.

Example 2.22. Let $f, g\colon A \to B$ be two functions. The coequalizer of f, g in **Set** is the quotient set $B/\!\sim$ where $\sim$ is the equivalence closure of the relation

$$\mathcal{R} = \{(f(a), g(a)) \mid a \in A\}.$$

The morphism of the coequalizer $k\colon B \to K$ is the quotient map, sending each $b \in B$ to its equivalence class $[b] \in B/\!\sim$. For more details, see Example 5.2.9 from [5].

For the next example we need some preliminaries. Recall that for every Hilbert space H and any set of vectors $S \subseteq H$, the *orthogonal complement* of S is

$$S^\perp = \{v \in H \mid \forall u \in S,\ \langle u|v\rangle = 0\}.$$

Since $S^\perp$ is a closed subspace of H, it follows that $S^\perp$ is a Hilbert space.

Example 2.23. Let $f, g \in \mathbf{FdHilb}(A, B)$. The coequalizer of f, g in **FdHilb** is $\mathrm{im}(f - g)^\perp$ where

$$\mathrm{im}(f - g) = \{b \in B \mid \exists a \in A,\ (f - g)(a) = b\}.$$

The morphism of the coequalizer $k\colon B \to \mathrm{im}(f - g)^\perp$ maps each vector in $\mathrm{im}(f - g)$ to 0 and each vector in $\mathrm{im}(f - g)^\perp$ to itself — the action of k on other vectors is obtained by linear extension.

2.5 The general case: limits and colimits

Products, terminal objects and equalizers all have a similar definition. If we extract their common pattern we reach the notion of a categorical limit, defined below.

Definition 2.24. Let $\mathbf{C}$ be a category. For an arbitrary diagram[7] in $\mathbf{C}$, let $\{A_i \in \mathbf{C}\}_{i \in I}$ be the set of objects in it. A *cone* is an object $X \in \mathbf{C}$ along a set of morphisms $\{h_i : X \to A_i\}_{i \in I}$ such that, when these are included in the diagram, it is satisfied for each $i \in I$ that every path from X to A_i yields the same morphism. A *limit* is a cone $\{s_i : L \to A_i\}_{i \in I}$ such that for any other cone $\{h_i : X \to A_i\}_{i \in I}$ there is a unique morphism $m : X \to L$ satisfying $h_i = s_i \circ m$ for each $i \in I$.

An equalizer of $f, g : A \to B$ is a limit of the diagram

$$A \underset{g}{\overset{f}{\rightrightarrows}} B$$

where a fork is just a particular kind of cone.[8] A product of two objects A and B is a limit of the diagram

$$A \qquad\qquad B$$

and a terminal object is a limit of the empty diagram.

We can define colimits dually; for the sake of completeness, the explicit definition is provided below.

Definition 2.25. Let $\mathbf{C}$ be a category. For an arbitrary diagram in $\mathbf{C}$, let $\{A_i \in \mathbf{C}\}_{i \in I}$ be the set of objects in it. A *cocone* is an object $X \in \mathbf{C}$ along a set of morphisms $\{h_i : A_i \to X\}_{i \in I}$ such that, when these are included in the diagram, it is satisfied for each $i \in I$ that every path from A_i to X yields the same morphism. A *colimit* is a cocone $\{s_i : A_i \to K\}_{i \in I}$ such that for any other cocone $\{h_i : A_i \to X\}_{i \in I}$ there is a unique morphism $m : K \to X$ satisfying $h_i = m \circ s_i$ for each $i \in I$.

As you may expect by now, an initial object is a colimit of the empty diagram, a coproduct is a colimit of a diagram with two objects and no morphisms and so on.

Proposition 2.26. *Let $L, L' \in \mathbf{C}$ where both L and L' are limits of the same diagram in $\mathbf{C}$. Then, there is an isomorphism $L \cong L'$.*

Proof. The proof follows the same argument as that of Proposition 2.7. For more details, see Proposition 3.1.7 from [6]. □

With the definitions of limit and colimit in hand, we can easily introduce the notion of pullbacks and pushouts, the last special case of limit that will be considered in this section.

Definition 2.27. Let $\mathbf{C}$ be a category and consider the following diagram in it:

$$\begin{array}{ccc} & & B \\ & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array}$$

A *pullback* is a limit of this diagram.

[7] A diagram is a directed graph where the vertices are objects in $\mathbf{C}$ and the arrows are morphisms in the direction determined by their type

[8] Notice that a cone is comprised of one morphism per object in the diagram — in this case, two: $h_A : X \to A$ and $h_B : X \to B$ — whereas our definition of fork only considered a morphism $X \to A$. However, $h_B = f \circ h_A = g \circ h_A$ is imposed by the definition of cone so, in this particular case, it is unnecessary to explicitly provide the value of h_B, which is why it was omitted when introducing forks.

Example 2.28. Let $A, B, B' \in \mathbf{Set}$, let $f\colon A \to B$ be a function and let $B' \subseteq B$. The inverse image $f^{-1}(B') = \{a \in A \mid f(a) \in B'\}$ is a pullback of the diagram

$$\begin{array}{ccc} & & B' \\ & & \big\downarrow \\ A & \xrightarrow[f]{} & B \end{array}$$

in **Set**.

Example 2.29. Let **CMon** $\hookrightarrow$ **Mon** and **Grp** $\hookrightarrow$ **Mon** be the canonical embedding functors introduced in the previous section. The category **Ab** is a pullback of the diagram

$$\begin{array}{ccc} & & \mathbf{Grp} \\ & & \big\downarrow \\ \mathbf{CMon} & \hookrightarrow & \mathbf{Mon} \end{array}$$

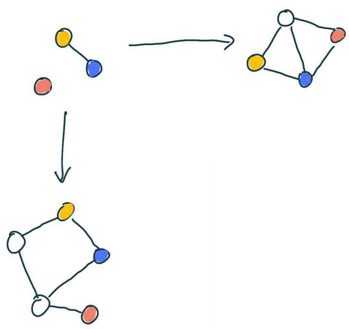
in **Cat**.

Definition 2.30. Let **C** be a category and consider the following diagram in it:

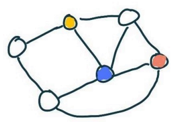
$$\begin{array}{ccc} C & \xrightarrow{g} & B \\ {\scriptstyle f}\big\downarrow & & \\ A & & \end{array}$$

A *pushout* is a colimit of this diagram.

Example 2.31. In the category **Graph** of graphs and graph homomorphisms, consider the following diagram



where morphisms map colored vertices to colored vertices. The following graph



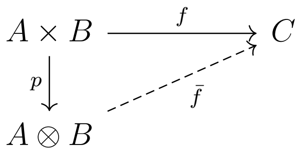
is a pushout of this diagram.

2.6 Final remarks

The universal property of the tensor product. Let $A, B, C \in \mathbf{FdHilb}$ and let $A \times B \in \mathbf{Set}$ be the Cartesian product of the underlying sets. A function $f \in \mathbf{Set}(A \times B, C)$ is *bilinear* if for all $a, a' \in A$ and $b, b' \in B$:

$$f(a + a', b) = f(a, b) + f(a', b)$$
$$f(a, b + b') = f(a, b) + f(a, b').$$

Let $\mathrm{Bilin}^{A\times B}(C)$ be the set of bilinear functions of type $A \times B \to C$. The tensor product $A \otimes B \in \mathbf{FdHilb}$ is a Hilbert space along with a bilinear function $p \in \mathrm{Bilin}^{A\times B}(A \otimes B)$ characterised by the following universal property: for each $C \in \mathbf{FdHilb}$ and each $f \in \mathrm{Bilin}^{A\times B}(C)$ there is a unique morphism $\bar{f} \in \mathbf{FdHilb}(A \otimes B, C)$ such that the following diagram

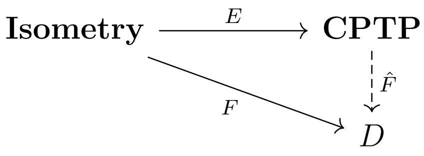


commutes in **Set**. Put another way, for each $C \in \mathbf{FdHilb}$ there is a bijection

$$\mathrm{Bilin}^{A\times B}(C) \cong \mathbf{FdHilb}(A \otimes B, C)$$

which allows us to unequivocally represent bilinear functions $A \times B \to C$ as linear maps $A \otimes B \to C$.

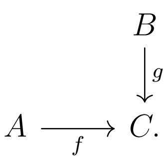
The universal property of CPTP [4]. Let **Isometry** be the category whose objects are finite dimensional Hilbert spaces and whose morphisms are isometries. Let **CPTP** be the category whose objects are finite dimensional Hilbert spaces and whose morphisms $A \to B$ are completely positive trace-preserving maps $\mathcal{L}(A) \to \mathcal{L}(B)$.[9] Let $E\colon \mathbf{Isometry} \to \mathbf{CPTP}$ be the functor acting as identity on objects and mapping each isometry V to its (pure) CPTP map $V(-)V^\dagger$. The following universal property has to do with monoidal categories, which are introduced in the next section. Let $(\mathbf{D}, \otimes, I)$ be a symmetric monoidal category whose monoidal unit I is a terminal object and let $F\colon \mathbf{Isometry} \to D$ be a symmetric monoidal functor. There is a unique symmetric monoidal functor $\hat{F}$ making the following diagram



commute in **Cat**. Loosely speaking, this characterises **CPTP** as the canonical completion obtained after adding a discarding map to **Isometry**. More details on this will be discussed later in the course.

2.7 Exercises

[!] **2.1.** Let **C** be a category and let the following be a diagram in it



[9] Where $\mathcal{L}(A)$ is the Hilbert space of linear maps of type $A \to A$.

Assume the product of A and B exists, and so does the equalizer of

$$A \times B \underset{g \circ \pi_B}{\overset{f \circ \pi_A}{\rightrightarrows}} C$$

Prove that an equalizer of this diagram is a pullback of the first diagram.

`[M]` **2.2.** Prove the claim of the following examples.

(a) Example 2.20.

(b) Example 2.23.

(c) Example 2.28.

`[!]` **2.3.** Let $\mathbf{C}$ be a category and let $A, B, C \in \mathbf{C}$ be objects in it.

(a) Assume the product of A and B exists and that the product of $A \times B$ and C exists as well. Prove that $(A \times B) \times C$ is a limit of the diagram

$$A \qquad B \qquad C$$

(b) Assume the product of B and C exists and that the product of A and $B \times C$ exists as well. Prove that $A \times (B \times C)$ is a limit of the diagram

$$A \qquad B \qquad C$$

(c) Conclude, using Proposition 2.26, that there is an isomorphism

$$(A \times B) \times C \cong A \times (B \times C).$$

`[M]` **2.4.** (Exercise 5.1.35 from [5]) Take a commutative diagram

$$\begin{array}{ccccc} \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\ \downarrow & & \downarrow & & \downarrow \\ \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \end{array}$$

in some category. Suppose that the right-hand square is a pullback. Show that the left-hand square is a pullback if and only if the outer rectangle is a pullback.

3 Monoidal categories

We will often find that 'sequential' composition of morphisms $\circ$ is not the only reasonable notion of composition. The notion of 'parallel' composition is formalised in monoidal categories.

We need some preliminaries before defining monoidal categories. Let $\mathbf{C}$ and $\mathbf{D}$ be categories; we can define a new category $\mathbf{C} \times \mathbf{D}$ whose objects are pairs (A, B) for each $A \in \mathbf{C}$ and $B \in \mathbf{D}$ and whose morphisms $(A, B) \to (C, D)$ are pairs (f, g) for each $f \in \mathbf{C}(A, C)$ and $g \in \mathbf{D}(B, D)$.[10]

Let $F \colon \mathbf{A} \times \mathbf{B} \to \mathbf{C}$ be a functor; then functoriality imposes

$$F(g, k) \circ_{\mathbf{C}} F(f, h) = F(g \circ_{\mathbf{A}} f, k \circ_{\mathbf{B}} h)$$

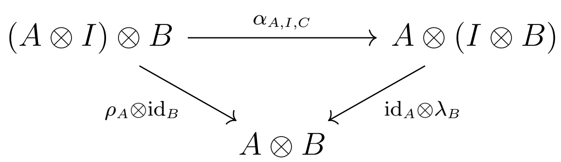
where f and g are morphisms in $\mathbf{A}$ and k and h are morphisms in $\mathbf{B}$. We tend to write these kind of 'binary functors' using operator notation. For instance, we may rename the functor F above as $\otimes$ and write $F(g, k)$ as $(g \otimes k)$ instead. Then, the previous equation becomes

$$(g \otimes k) \circ_{\mathbf{C}} (f \otimes h) = (g \circ_{\mathbf{A}} f) \otimes (k \circ_{\mathbf{B}} h). \tag{4}$$

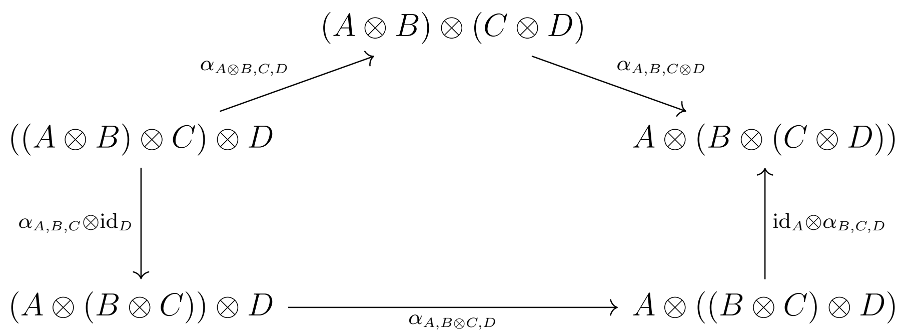
Definition 3.1. Let $\mathbf{C}$ be a category, let $\otimes \colon \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ be a functor and let I be an object in $\mathbf{C}$. The triple $(\mathbf{C}, \otimes, I)$ is a *monoidal category* if there is:

- a natural isomorphism α known as the *associator* whose components are morphisms in $\mathbf{C}$ of type
$$\alpha_{A,B,C} \colon (A \otimes B) \otimes C \to A \otimes (B \otimes C)$$
for each $A, B, C \in \mathbf{C}$;
- a natural isomorphism λ known as the *left unitor* whose components are morphisms in $\mathbf{C}$ of type
$$\lambda_A \colon I \otimes A \to A$$
for each $A \in \mathbf{C}$;
- a natural isomorphism ρ known as the *right unitor* whose components are morphisms in $\mathbf{C}$ of type
$$\rho_A \colon A \otimes I \to A$$
for each $A \in \mathbf{C}$;

such that the diagrams below commute for all choices of objects $A, B, C, D \in \mathbf{C}$. We refer to $\otimes$ as the *monoidal product* and I as the *monoidal unit*.



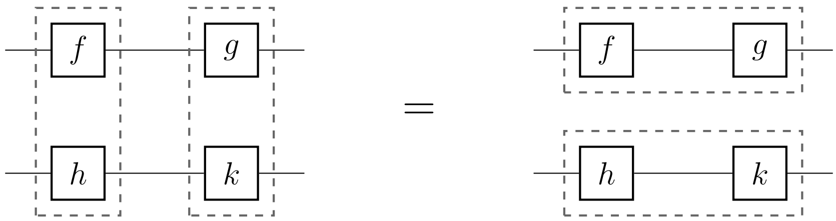
[10]This happens to be a product in the category **Cat**!



The first of the commuting diagrams above (the triangle) imposes that the object I acts as the unit of 'parallel' composition $\otimes$, whereas the second commuting diagram (the pentagon) imposes that $\otimes$ is associative. As discussed above, the requirement that $\otimes \colon \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ is a functor imposes an interchange law between $\circ$ and $\otimes$:

$$(g \otimes k) \circ (f \otimes h) = (g \circ f) \otimes (k \circ h). \tag{5}$$

The intuition behind this equation is provided in the picture below — it does not matter whether we first compose in parallel or sequentially, the result is the same.



By convention, morphisms are depicted as labelled boxes, connected via wires when composed sequentially (using $\circ$) and drawn one on top of the other when composed in parallel. Some examples of monoidal categories are presented below.

Example 3.2. Let $\times \colon \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$ be the functor acting on objects as the Cartesian product of sets. On morphisms $f \colon A \to C$ and $g \colon B \to D$, the functor yields a function $f \times g$ that maps each $(a, b) \in A \times B$ to $(f(a), g(b)) \in C \times D$. Let $\{\bullet\}$ be an arbitrary singleton set; then, $(\mathbf{Set}, \times, \{\bullet\})$ is a monoidal category. Its associators and unitors are the unique morphisms of their type.

Example 3.3. Let $\oplus \colon \mathbf{Rel} \times \mathbf{Rel} \to \mathbf{Rel}$ be the functor acting on objects as the disjoint union of sets $A \oplus B = A \uplus B$. On morphisms $\mathcal{R} \colon A \to C$ and $\mathcal{S} \colon B \to D$, the functor yields the following relation:

$$\mathcal{R} \oplus \mathcal{S} = \{(a_{\text{left}}, c_{\text{left}}) \mid a \mathcal{R} c\} \cup \{(b_{\text{right}}, d_{\text{right}}) \mid b \mathcal{S} d\}.$$

It can be shown that $(\mathbf{Rel}, \oplus, \varnothing)$ is a monoidal category. Its associators and unitors are the unique morphisms of their type.

Example 3.4. Let $\oplus \colon \mathbf{FdHilb} \times \mathbf{FdHilb} \to \mathbf{FdHilb}$ be the functor acting as the direct sum of Hilbert spaces and linear maps and let $\{0\}$ be the zero-dimensional vector space. It can be shown that $(\mathbf{FdHilb}, \oplus, \{0\})$ is a monoidal category. Its associators and unitors are the unique morphisms of their type.

All of these examples have something in common: the monoidal product on objects is a categorical product and the monoidal unit is a terminal object. Indeed, every category with products and terminal objects can be given a monoidal structure; we will look into this in Exercise 3.1. The proof of each of the examples above can be derived immediately from the claim in that exercise. In contrast, the monoidal product of the following examples does not come from a categorical product.

Example 3.5. Let $\otimes\colon \mathbf{Rel} \times \mathbf{Rel} \to \mathbf{Rel}$ be the functor acting on objects as Cartesian product of sets $A \otimes B = A \times B$. On morphisms $\mathcal{R}\colon A \to C$ and $\mathcal{S}\colon B \to D$, the functor yields the following relation:

$$(a,b)\mathcal{R}{\otimes}\mathcal{S}(c,d) \iff a\mathcal{R}c \text{ and } b\mathcal{S}d.$$

It can be shown that $(\mathbf{Rel}, \otimes, \{\bullet\})$ is a monoidal category where:

- the associator $\alpha_{A,B,C}$ is the relation $((a,b),c) \sim (a,(b,c))$ for each $a \in A$, $b \in B$ and $c \in C$,
- the left unitor λ_A is the relation $(\bullet, a) \sim a$ for each $a \in A$ and
- the right unitor ρ_A is the relation $(a, \bullet) \sim a$ for each $a \in A$.

Example 3.6. Let $\otimes\colon \mathbf{FdHilb} \times \mathbf{FdHilb} \to \mathbf{FdHilb}$ be the functor acting as the tensor product on vector spaces and linear maps. It can be shown that $(\mathbf{FdHilb}, \otimes, \mathbb{C})$ is a monoidal category where:

- the associator $\alpha_{A,B,C}$ is the unique linear map such that $(a{\otimes}b){\otimes}c \mapsto a{\otimes}(b{\otimes}c)$ for each $a \in A$, $b \in B$ and $c \in C$,
- the left unitor λ_A is the unique linear map such that $1 \otimes a \mapsto a$ for each $a \in A$ and
- the right unitor ρ_A is the unique linear map $a \otimes 1 \mapsto a$ for each $a \in A$.

Example 3.7. Let $\oplus\colon \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$ be the functor acting on objects as disjoint union $A \oplus B = A \uplus B$. On morphisms $f\colon A \to C$ and $g\colon B \to D$, the funtor yields the following function:

$$f{\oplus}g(x) = \begin{cases} f(a) & \text{if} \quad x = a_{\text{left}} \\ g(b) & \text{if} \quad x = b_{\text{right}}. \end{cases}$$

It can be shown that $(\mathbf{Set}, \oplus, \varnothing)$ is a monoidal category where:

- the associator $\alpha_{A,B,C}$ is the function mapping $(a_{\text{left}})_{\text{left}} \mapsto a_{\text{left}}$ for each $a \in A$, mapping $(b_{\text{right}})_{\text{left}} \mapsto (b_{\text{left}})_{\text{right}}$ for each $b \in B$ and mapping $c_{\text{right}} \mapsto (c_{\text{right}})_{\text{right}}$ for each $c \in C$,
- the left unitor λ_A is the function mapping $a_{\text{right}} \mapsto a$ for each $a \in A$ and
- the right unitor ρ_A is the function mapping $a_{\text{left}} \mapsto a$ for each $a \in A$.

You should convince yourself that (5) holds for the monoidal product in each of this examples, and check that the pentagon and triangle diagrams from Definition 3.1 do commute. You should also check that α, λ and ρ are natural isomorphisms. Since associators and unitors are given explicitly in these last three examples, all these checks can be done by direct calculation. Notice that, in the case of $(\mathbf{Set}, \oplus, \varnothing)$, we have defined the monoidal structure using a coproduct and an initial object in $\mathbf{Set}$.

3.1 Monoidal functors

Along each new flavour of categories comes a refined notion of structure-preserving functor.

Definition 3.8. Let $(\mathbf{C}, \otimes, I)$ and $(\mathbf{D}, \otimes', I')$ be monoidal categories and let $F\colon \mathbf{C} \to \mathbf{D}$ be a functor. Let μ be a natural isomorphism with components

$$\mu_{A,B}\colon F(A \otimes_{\mathbf{C}} B) \to F(A) \otimes_{\mathbf{D}} F(B)$$

and let $\mu_0\colon F(I_{\mathbf{C}}) \to I_{\mathbf{D}}$ be an isomorphism in $\mathbf{D}$. We say F is a monoidal functor if the following diagrams commute for all $A, B, C \in \mathbf{C}$.

$$\begin{array}{ccc}
F((A \otimes B) \otimes C) & \xrightarrow{F(\alpha_{A,B,C})} & F(A \otimes (B \otimes C)) \\
\downarrow{\scriptstyle \mu_{A\otimes B,C}} & & \downarrow{\scriptstyle \mu_{A,B\otimes C}} \\
F(A \otimes B) \otimes' F(C) & & F(A) \otimes' F(B \otimes C) \\
\downarrow{\scriptstyle \mu_{A,B}\otimes'\mathrm{id}_{F(C)}} & & \downarrow{\scriptstyle \mathrm{id}_{F(A)}\otimes'\mu_{B,C}} \\
(F(A) \otimes' F(B)) \otimes' F(C) & \xrightarrow[\alpha'_{F(A),F(B),F(C)}]{} & F(A) \otimes' (F(B) \otimes' F(C))
\end{array}$$

$$\begin{array}{ccc}
F(A \otimes I) & \xrightarrow{\mu_{A,I}} & F(A) \otimes' F(I) \\
\downarrow{\scriptstyle F(\rho_A)} & & \downarrow{\scriptstyle \mathrm{id}_{F(A)}\otimes'\mu_0} \\
F(A) & \xleftarrow[\rho'_{F(A)}]{} & F(A) \otimes' I'
\end{array}
\qquad
\begin{array}{ccc}
F(I \otimes A) & \xrightarrow{\mu_{I,A}} & F(I) \otimes' F(A) \\
\downarrow{\scriptstyle F(\lambda_A)} & & \downarrow{\scriptstyle \mu_0\otimes'\mathrm{id}_{F(A)}} \\
F(A) & \xleftarrow[\lambda'_{F(A)}]{} & I' \otimes' F(A).
\end{array}$$

Example 3.9. Consider the monoidal categories $(\mathbf{FinSet}, \times, \{\bullet\})$ and $(\mathbf{FdHilb}, \otimes, \mathbb{C})$ from the previous examples — where **FinSet** is the subcategory of **Set** whose objects are all finite sets. There is a monoidal functor $F\colon \mathbf{FinSet} \to \mathbf{FdHilb}$ that maps each set $A \in \mathbf{FinSet}$ to the Hilbert space $F(A) \in \mathbf{FdHilb}$ spanned by taking A as a basis. For each morphism $f \in \mathbf{FinSet}(A, B)$, the functor yields a linear map $F(f) \in \mathbf{FdHilb}(F(A), F(B))$ defined by linear extension of f. Notice that $F(A \times B) \cong F(A) \otimes F(B)$ since both have the same dimension $|A \times B| = |A| \cdot |B|$; we define $\mu_{A,B}$ to be this isomorphism. Similarly, $F(\{\bullet\}) \cong \mathbb{C}$ provides μ_0.

Example 3.10. Consider the monoidal categories $(\mathbf{FinSet}, \oplus, \varnothing)$ and $(\mathbf{FdHilb}, \oplus, \{0\})$ from the previous examples — where **FinSet** is the subcategory of **Set** whose objects are all finite sets. There is a monoidal functor $F\colon \mathbf{FinSet} \to \mathbf{FdHilb}$ that maps each set $A \in \mathbf{FinSet}$ to the Hilbert space $F(A) \in \mathbf{FdHilb}$ spanned by taking A as a basis. For each morphism $f \in \mathbf{FinSet}(A, B)$, the functor yields a linear map $F(f) \in \mathbf{FdHilb}(F(A), F(B))$ defined by linear extension of f. Notice that $F(A \oplus B) \cong F(A) \oplus F(B)$ since both have the same dimension $|A \uplus B| = |A| + |B|$; we define $\mu_{A,B}$ to be this isomorphism. Similarly, $F(\varnothing) \cong \{0\}$ provides μ_0.

In these examples, each μ has not been given explicitly, but we know they exist. In principle, this is not enough to conclude that these functors are monoidal: we would still need to check that the diagrams from Definition 3.8 commute. Proving this is not trivial, but it isn't too difficult either — notice that $(\mathbf{FinSet}, \times, \{\bullet\})$ is built around a categorical product $\times$ and $(\mathbf{FdHilb}, \oplus, \{0\})$ is built around a coproduct $\oplus$.

3.2 The coherence theorem

We will now take a detour to justify the use and rigor of graphical language in monoidal categories. This section is quite informal; for a more in-depth treatment of the topic see Sections 1.3.3 and 1.3.4 from [3].

Let $(\mathbf{C}, \otimes, I)$ be a monoidal category and $A, B, C \in \mathbf{C}$ objects in it. The associator makes objects $(A \otimes B) \otimes C$ and $A \otimes (B \otimes C)$ isomorphic, but they are not necessarily the same. Indeed, in $(\mathbf{Set}, \times, \{\bullet\})$ the set $(A \times B) \times C$ and the set $A \times (B \times C)$ are not quite the same since $((a, b), c) \neq (a, (b, c))$. However, it can be argued that the difference between these simply comes down to mathematical fluff: $((a, b), c)$ is not quite the same element as $(a, (b, c))$ just due to how its data is wrapped differently within parentheses. In a strict monoidal gategory (defined below) these $(A \otimes B) \otimes C$ and $A \otimes (B \otimes C)$ are in fact the same object for all $A, B, C \in \mathbf{C}$.

Definition 3.11. A *strict monoidal category* is a monoidal category whose associators and unitors are all identities.

Example 3.12. Let $\mathbf{Mat}_{\mathbb{C}}$ be the category whose objects are the natural numbers and whose morphisms $n \to m$ are m-by-n matrices of complex numbers. Composition corresponds to matrix multiplication, whereas monoidal product $\otimes$ on objects corresponds to multiplication $n \otimes m = nm$ and on morphisms $f \otimes g$ corresponds to Kronecker product. Then, $(\mathbf{Mat}_{\mathbb{C}}, \otimes, 1)$ is a strict monoidal category.

Notice that $(n \otimes m) \otimes k$ is the *same* as $n \otimes (m \otimes k)$ thanks to multiplication of natural numbers being associative. Similarly, $(f \otimes g) \otimes h$ is the *same* matrix as $f \otimes (g \otimes h)$ thanks to the Kronecker product being associative. Similar checks follow for the unitors so that, indeed, $(\mathbf{Mat}_{\mathbb{C}}, \otimes, 1)$ is a strict monoidal category.

Even though the monoidal category $(\mathbf{FdHilb}, \otimes, \mathbb{C})$ and the strict monoidal category $(\mathbf{Mat}_{\mathbb{C}}, \otimes, 1)$ are not quite the same, they are definitely closely related. **FdHilb** is not strict — once again, $(u, (v, w))$ and $((u, v), w)$ are not quite the same vectors — nevertheless, there is a monoidal functor $(\mathbf{FdHilb}, \otimes, \mathbb{C}) \to (\mathbf{Mat}_{\mathbb{C}}, \otimes, 1)$ sending each linear map to its matrix representation and sending each Hilbert space $A \in \mathbf{FdHilb}$ to the natural number $\dim(A)$. Such a functor is special in that it is an instance of an equivalence between categories.

Definition 3.13. Let $F \colon \mathbf{C} \to \mathbf{D}$ be a functor.

- We say that F is *faithful* if the action of F on morphisms $\mathbf{C}(A, B) \to \mathbf{D}(F(A), F(B))$ is injective for each $A, B \in \mathbf{C}$.
- We say that F is *full* if the action of F on morphisms $\mathbf{C}(A, B) \to \mathbf{D}(F(A), F(B))$ is surjective for each $A, B \in \mathbf{C}$.
- We say that F is *essentially surjective* if for every object $B \in \mathbf{D}$ there is an object $A \in \mathbf{C}$ such that $B \cong F(A)$.

We say that F is an *equivalence* if it is full, faithful and essentially surjective. When $\mathbf{C}$ and $\mathbf{D}$ are monoidal categories and F is an equivalence, we say that F is a *monoidal equivalence* if F is a monoidal functor.

It is immediate to check that the functor $\mathbf{FdHilb} \to \mathbf{Mat}_{\mathbb{C}}$ we were discussing above is an equivalence. A bit more work is required to prove that it is a monoidal equivalence, but this is indeed the case. When equipped with an equivalence $F \colon \mathbf{C} \to$

$\mathbf{D}$, we can prove that two morphisms $f, g\colon A \to B$ in $\mathbf{C}$ are the same by checking whether their image $F(f), F(g)$ are the same — in fact, we only require faithfulness for this. Indeed, we are used to saying that two linear maps are the same if and only if their matrix representation is the same.

Proposition 3.14. *Let $F\colon \mathbf{C} \to \mathbf{D}$ be a faithful functor and let $f, g\colon A \to B$ be morphisms in $\mathbf{C}$. Then:*

$$f = g \iff F(f) = F(g)$$

Proof. The ($\Rightarrow$) direction follows trivially and the ($\Leftarrow$) direction follows from the action of F on $\mathbf{C}(A, B) \to \mathbf{D}(F(A), F(B))$ being injective. □

Moreover, when $F\colon \mathbf{C} \to \mathbf{D}$ is an equivalence we can define a functor $G\colon \mathbf{D} \to \mathbf{C}$ that is also an equivalence and acts roughly[11] as the inverse of F. Hence, an equivalence $F\colon \mathbf{C} \to \mathbf{D}$ not only satisfies Proposition 3.14 but also guarantees the existence of another functor on the opposite direction $G\colon \mathbf{D} \to \mathbf{C}$ satisfying Proposition 3.14. Informally, we can think of an equivalence between categories as stating that "if an equation holds in $\mathbf{C}$ it also holds in $\mathbf{D}$ and vice versa".

With this knowledge at hand, let's look at the strictification theorem.

Theorem 3.15. *Every monoidal category is monoidally equivalent to a strict monoidal category.*

Informally, this means that given *any* monoidal category — no matter how complicated its definition is — there is a strict monoidal category where the mathematical fluff has been omitted. Then, manipulation of expressions becomes easier while guaranteeing that any calculation performed in the simpler category yields a correct calculation in the original one.

However, the strictification theorem has two important caveats.

- Even though it *does* tell you how to construct the strict monoidal category corresponding to your original monoidal category, the construction is so abstract that it isn't particularly useful for calculations. In particular, applying the strictification theorem to **FdHilb** does not yield something as concise as $\mathbf{Mat}_{\mathbb{C}}$ but rather some abstract nonesense that captures the essence of $\mathbf{Mat}_{\mathbb{C}}$ if you squint at it for long enough. Thus, we should take the theorem as a statement about the existence of an equivalent strict monoidal category rather than a method for constructing it.

- Even if you ended up with a manageable strict category such as $\mathbf{Mat}_{\mathbb{C}}$, much of the details from the original category have been lost. For instance, the functor $\mathbf{FdHilb} \to \mathbf{Mat}_{\mathbb{C}}$ maps all Hilbert spaces of dimension n to the same object in $\mathbf{Mat}_{\mathbb{C}}$. If you are given a matrix without any context, it is just a bag of numbers to you; at the very least you need to know which are the basis vectors: are they the computational basis? the X basis? are the basis vectors coordinates in space? are they wave functions? Working with the strict category alone is just not enough: you make calculations in the strict category $\mathbf{Mat}_{\mathbb{C}}$ but assign meaning to them via the functor $\mathbf{FdHilb} \to \mathbf{Mat}_{\mathbb{C}}$.

What is the point of the strictification theorem then? Possibly its most important purpose is that it acts as a stepping stone for the coherence theorem, stated below.

[11] It is only a proper inverse if the action of F on objects is a bijection; being essentially surjective is not enough. However, let's not go down that rabbit hole.

Theorem 3.16. *Let* $(\mathbf{C}, \otimes, I)$ *be a monoidal category. Let* f *and* g *be two morphisms in* $\mathbf{C}$ *built exclusively from* α, α^{-1}, λ, λ^{-1}, ρ, ρ^{-1} *and* id *by combining them using* $\circ$ *and* $\otimes$. *If* f *and* g *have the same type then* $f = g$.

Sketch. Let $\mathbf{D}$ be the strict monoidal category given by the strictification theorem and let $F : \mathbf{C} \to \mathbf{D}$ be its monoidal equivalence. Recall that $F(- \circ -) = F(-) \circ F(-)$ and $F(- \otimes -) \cong F(-) \otimes F(-)$; apply these inductively to $F(f)$ and $F(g)$. Since $\mathbf{D}$ is strict, $F(\alpha) = \mathrm{id}$ and similarly with all other components of f and g. Since $\mathrm{id} \circ \mathrm{id} = \mathrm{id}$ and $\mathrm{id} \otimes \mathrm{id} = \mathrm{id}$ it follows that $F(f) = \mathrm{id} = F(g)$. But F is a faithful functor — it is an equivalence — so $f = g$ (direct from faithfulness, see Proposition 3.14). □

With the coherence theorem we get the best of both worlds. On one hand, we know we can safely ignore associators and unitors in our equations as long as everything "type checks", reducing unnecessary mathematical fluff. On the other hand, we never leave our original category, so the meaning of our objects and morphisms remains intact.

A natural question arises: if we can disregard associators and unitors, why do we even bother defining them? This is a misconception that arises from looking at the problem backwards. In practice, the usual workflow is the following:

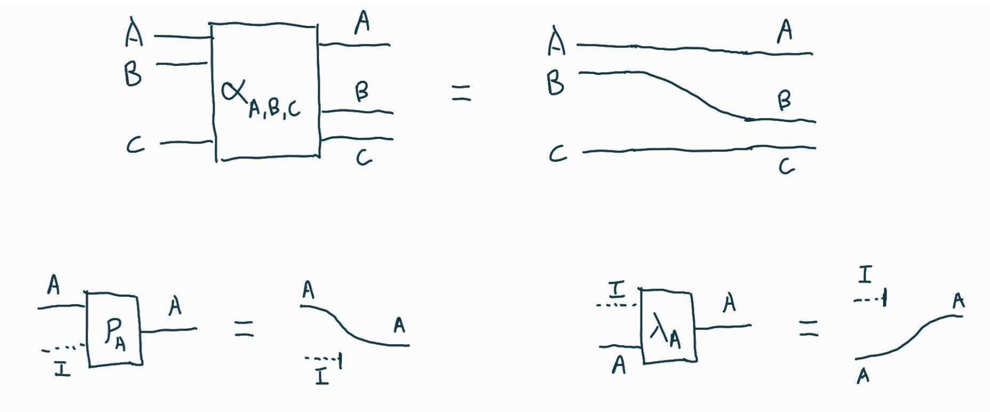
1. Pick a category $\mathbf{C}$ and come up with a functor $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$. Then, you ask yourself whether $\otimes$ can be used to give a monoidal structure to $\mathbf{C}$.

2. To answer this, you must come up with associators and unitors and verify that the triangle and pentagon equations from Definition 3.1 hold. In some sense, these guarantee that $\otimes$ is a well-behaved 'parallel composition'.

3. Once you know that $(\mathbf{C}, \otimes, I)$ is monoidal you no longer need to worry about associators and unitors, thanks to the coherence theorem. However, the step of verifying the pentagon and triangle equations was essential — you had to check that $\otimes$ was well-behaved enough for the coherence theorem to hold.

This is useful when working with categories whose monoidal product is complex. In the case of $(\mathbf{FdHilb}, \otimes, \mathbb{C})$, the monoidal product $\otimes$ is just the usual tensor product of vector spaces. However, when working with the category **Hilb** of arbitrary Hilbert spaces (including infinite dimensional ones) and bounded linear maps, the usual tensor product of Hilbert spaces is *not* a Hilbert space; we need to 'add' some more vectors for it to satisfy completeness. Then, proving that $(\mathbf{Hilb}, \otimes, \mathbb{C})$ is a monoidal category is not so trivial anymore. The good news is that, once we prove it, we no longer need to worry about the monoidal structure: all works transparently thanks to the coherence theorem. Similarly, one may think that there is a well-behaved notion of tensor product of topological abelian groups in some appropriate category **TopAb**. However, when you try to formalise it as a monoidal product you find out that the pentagon equation fails, so it does not provide a monoidal structure for **TopAb**, preventing you from using the coherence theorem.

Since we can always disregard associators and unitors as long as everything 'type checks', we may remove all appearances of α, λ and ρ from our equations — as long as we have a system to keep track of where the objects are in relation to each other (*i.e.* keep track of the type). This is precisely what the graphical language of monoidal categories does:



This, along with the graphical language's 'native' distinction between sequential composition $\circ$ and parallel composition $\otimes$ — which removes the need for parentheses when combining these two (see (5) and the corresponding diagram) — makes the graphical language of string diagrams a concise and intuitive language for equation manipulation. And, thanks to the coherence theorem, it is fully rigorous and usable in all monoidal categories, no matter how complicated they are.

Something essential to keep in mind is that — in a similar manner $\mathbf{Mat}_{\mathbb{C}}$ alone lacks meaning and must come along a functor $\mathbf{FdHilb} \to \mathbf{Mat}_{\mathbb{C}}$ providing its semantics — playing with diagrams lacks meaning unless we understand what each of its boxes 'mean'. As usual, we can capture the meaning of diagrams via a functor $F\colon \mathbf{Diagram} \to \mathbf{C}$. Here, **Diagram** is the category whose objects are type signatures and whose morphisms are diagrams, whereas F maps each diagram to its intended interpretation as a morphism in $\mathbf{C}$. This is a common theme in computer science: **Diagram** provides the syntax, F provides the semantics.

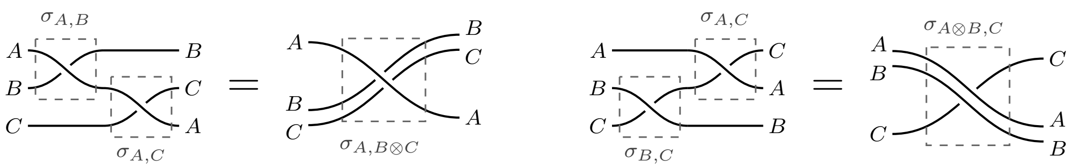
3.3 Symmetric monoidal categories

Let's add more flavour to our monoidal categories.

Definition 3.17. Let $(\mathbf{C}, \otimes, I)$ be a monoidal category and let σ be a natural isomorphism with components

$$\sigma_{A,B}\colon A \otimes B \to B \otimes A.$$

We say $\mathbf{C}$ is a *braided monoidal category* if σ satisfies the axioms given below. We refer to σ as braiding and represent it graphically as the crossing of wires; the axioms are represented graphically as:



We say $\mathbf{C}$ is a *symmetric monoidal category* if, additionally, $\sigma_{A,B}^{-1} = \sigma_{B,A}$ for all pairs of objects.

Definition 3.18. A *braided monoidal functor* is a monoidal functor $F\colon \mathbf{C} \to \mathbf{D}$ between braided monoidal categories such that the diagram

$$\begin{array}{ccc} F(A \otimes B) & \xrightarrow{F(\sigma_{A,B})} & F(B \otimes A) \\ {\scriptstyle \mu_{A,B}}\downarrow & & \downarrow{\scriptstyle \mu_{B,A}} \\ F(A) \otimes' F(B) & \xrightarrow[\sigma_{F(A),F(B)}]{} & F(B) \otimes' F(A) \end{array}$$

commutes, where σ is the braiding in $\mathbf{C}$, σ' is the braiding in $\mathbf{D}$ and μ is the natural isomorphism that makes F a monoidal functor. A *symmetric monoidal functor* is a braided monoidal functor between symmetric monoidal categories.

Both **Set** and **Vect** are symmetric monoidal categories with either of the monoidal structures discussed in the previous examples. Both monoidal functors **FinSet** $\to$ **FdHilb** discussed in previous examples are symmetric monoidal functors.

There are many other flavours of categories, each of them adding more structure by introducing new operations and axioms. In the next section we will introduce two more categories of relevance to the study of quantum processes: dagger categories and compact closed categories. At its core, category theory is the classification of different flavours of categories, the description of how different categories are related to each other (via functors) and the study of how general properties (often *universal* properties) arise from the abstract structure that is imposed.

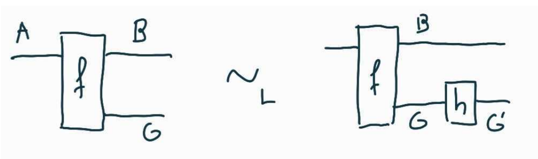
3.4 Exercises

[!] **3.1.** Let $\mathbf{C}$ be a category with products $A \times B$ for each $A, B \in \mathbf{C}$ and a terminal object I. Prove that $(\mathbf{C}, \times, I)$ is a symmetric monoidal category. **Hint**: Use Exercise 2.1 to define the associator. Use uniqueness of the universal morphism of limits to prove the pentagon equation. A similar approach will work for the unitors and braiding as well.

[!] **3.2.** Let $(\mathbf{C}, \otimes, I)$ be a symmetric monoidal category; let $f\colon A \to B \oplus G$ and $f'\colon A \to B \oplus G'$ be two morphisms in $\mathbf{C}$ and define the following relation:

$$f' \leq_L f \iff \exists h \in \mathbf{C}(G, G'), f' = (\mathrm{id} \otimes h) \circ f. \tag{6}$$

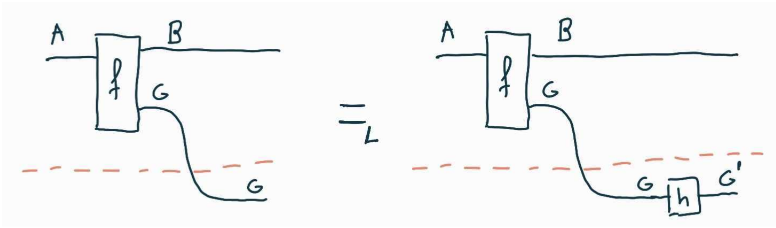
Let $\sim_L$ be the equivalence closure[12] of $\leq_L$.



The objects G and G' are not special in any way, they are just arbitrary objects of $\mathbf{C}$ as A and B are. Let $L[\mathbf{C}]$ be the category whose objects are the same as $\mathbf{C}$ and whose morphisms $A \to B$ are equivalence classes generated by $\sim_L$ on morphisms $A \to B \otimes -$ in $\mathbf{C}$. We may represent a morphism $f \in L[\mathbf{C}](A, B)$ graphically by taking advantage of the string diagram language of $\mathbf{C}$, where we separate the objects

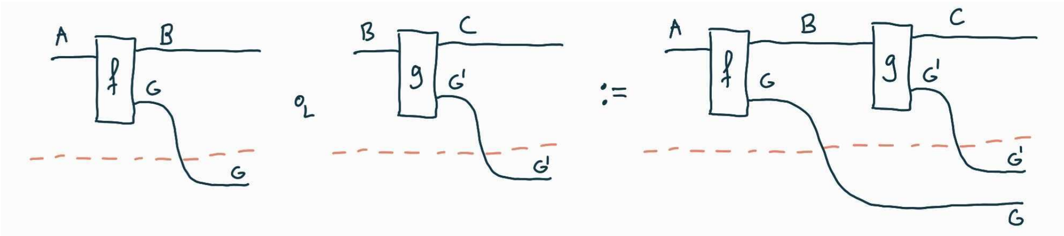
[12] The equivalence closure of a relation $\mathcal{R}$ is the smallest equivalence relation containing $\mathcal{R}$.

that are being quotiented over by $\sim_L$ with a dashed red line. These following two morphisms are *the same* in $L[\mathbf{C}]$ and their type is $A \to B$:



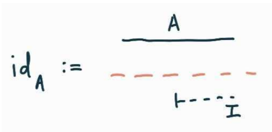
.

Let $f \in L[\mathbf{C}](A, B)$ and $g \in L[\mathbf{C}](B, C)$; composition in $L[\mathbf{C}]$ is defined as follows:



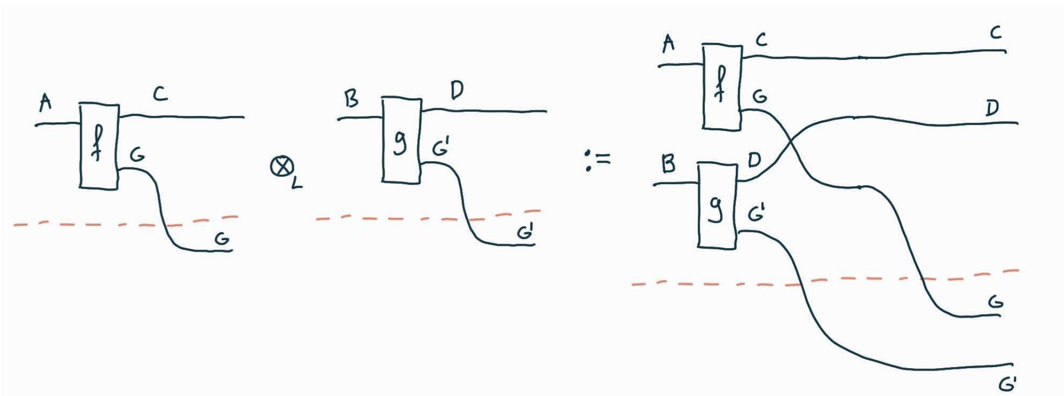
.

For each object $A \in L[\mathbf{C}]$, its identity is defined as follows:



.

Let $f \in L[\mathbf{C}](A, C)$ and $g \in L[\mathbf{C}](B, D)$; monoidal product in $L[\mathbf{C}]$ is defined as follows:



.

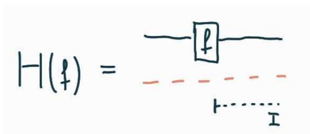
The monoidal unit in $L[\mathbf{C}]$ is I, the same as in $\mathbf{C}$.

(a) Prove that $L[\mathbf{C}]$ is a category (see Definition 1.3).

(b) Prove that $(L[\mathbf{C}], \otimes_L, I)$ is a symmetric monoidal category (see Definition 3.1). **Hint**: you must come up with a definition of associators and unitors in $L[\mathbf{C}]$ using those in $\mathbf{C}$.

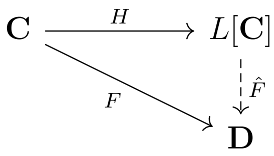
(c) Prove that I is a terminal object in $L[\mathbf{C}]$ (see Definition 2.14).

(d) Prove that there is a symmetric monoidal functor $H \colon \mathbf{C} \to L[\mathbf{C}]$ that acts as the identity on objects and acts as follows on morphisms (see Definitions 1.9 and 3.8):



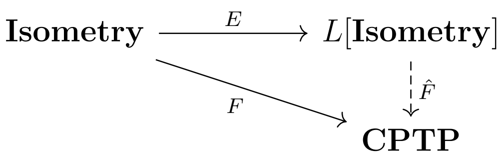
.

(e) Let $(\mathbf{D}, \otimes', I')$ be a symmetric monoidal category whose monoidal unit I' is terminal. Let $F\colon \mathbf{C} \to \mathbf{D}$ be a symmetric monoidal functor. Prove that there is a unique symmetric monoidal functor $\hat{F}\colon L[\mathbf{C}] \to \mathbf{D}$ such that the following diagram commutes. **Hint**: start by attempting to come up with a symmetric monoidal functor $\hat{F}$ that makes the diagram commute; then, check that every choice you made when defining it was the unique valid choice; this will imply uniqueness.



`[Q]` **3.3.** Notice that the last task in the previous exercise characterises $L[\mathbf{C}]$ via a universal property. In this exercise we look at what $L[\mathbf{Isometry}]$ is. Let **Isometry** be the category whose objects are finite dimensional Hilbert spaces and whose morphisms are isometries. Let **CPTP** be the category whose objects are finite dimensional Hilbert spaces and whose morphisms $A \to B$ are completely positive trace-preserving maps $\mathcal{L}(A) \to \mathcal{L}(B)$, where $\mathcal{L}(A)$ is the Hilbert space of linear maps of type $A \to A$. Let $E\colon \mathbf{Isometry} \to \mathbf{CPTP}$ be the functor that acts as the identity on objects and maps isometries $V \in \mathbf{Isometry}(A, B)$ to CPTP maps $V(-)V^\dagger$.

(a) Let $(\mathbf{D}, \otimes', I')$ be a symmetric monoidal category whose monoidal unit I' is terminal. Let $F\colon \mathbf{Isometry} \to \mathbf{D}$ be a symmetric monoidal functor. Prove — using the Stinespring's dilation theorem — that there is a unique symmetric monoidal functor $\hat{F}\colon L[\mathbf{Isometry}] \to \mathbf{CPTP}$ such that the following diagram commutes. **Hint**: start by attempting to come up with a symmetric monoidal functor $\hat{F}$ that makes the diagram commute; then, check that every choice you made when defining it was forced — *i.e.* there was only one choice — this will imply uniqueness.



(b) Prove — using an argument similar to Proposition 2.7 — that there is a full and faithful functor $L[\mathbf{Isometry}] \to \mathbf{CPTP}$. Conclude that $L[\mathbf{Isometry}]$ and **CPTP** are equivalent (see Definition 3.13).

Discussion: we can conclude that CPTP maps and 'isometries with hiding' are equally valid formalism to describe open quantum processes. This exercise has been extracted from papers [4] and [2].

4 Dagger compact closed categories

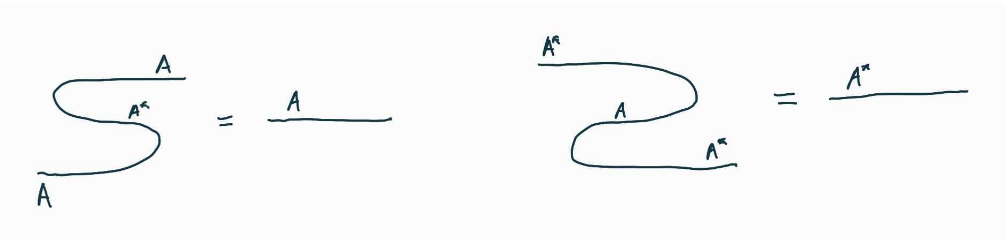
In this section we are going to formalise in category-theoretic terms two important properties of Hilbert spaces and their linear maps. The first one is captured by compact closed categories and we use it to show that the set of linear maps **FdHilb**(A, B) can be given the structure of a Hilbert space. The second one is captured by dagger categories and relates to the importance of adjoints of linear maps.

The notation and definitions in this section are taken for the most part from [7]. To learn more about dagger compact closed categories and their role in quantum computing, see [3].

4.1 Compact closed categories

Definition 4.1. Let $(\mathbf{C}, \otimes, I)$ be a symmetric monoidal category. For each $A \in \mathbf{C}$ let there be another object $A^* \in \mathbf{C}$ and morphisms $\eta_A : I \to A \otimes A^*$ and $\epsilon_A : A^* \otimes A \to I$ depicted in the graphical calculus as follows:

We say such a category $\mathbf{C}$ is a *compact closed category* if the following equations are satisfied for all $A \in \mathbf{C}$:



(7)

Example 4.2. We already know from previous examples that $(\mathbf{Rel}, \otimes, \{\bullet\})$ is a symmetric monoidal category. For each set $A \in \mathbf{Rel}$ let $A^* = A$ and define

$$\eta_A = \{(\bullet, (a, a)) \mid a \in A\}$$
$$\epsilon_A = \{((a, a), \bullet) \mid a \in A\}.$$

It is straightforward to check that **Rel** is a compact closed category under these definitions.

Example 4.3. We already know from previous examples that $(\mathbf{FdHilb}, \otimes, \mathbb{C})$ is a symmetric monoidal category. For each finite dimensional Hilbert space $A \in \mathbf{FdHilb}$ let A^* be the Hilbert space of functionals $A \to \mathbb{C}$; *i.e.* in bra-ket notation:

$$A^* = \{\langle a| \mid a \in A\}.$$

It is a standard exercise in linear algebra to prove that A^* is indeed in **FdHilb**.[13] For each $A \in \mathbf{FdHilb}$ let $\mathcal{B}_A$ be an orthonormal basis and let $\eta_A : I \to A \otimes A^*$ and

[13] Addition, scalar multiplication and inner product can be defined directly from those of A via $\langle x| + \langle y| = \langle x + y|$, etc. Completeness follows from the fact that A^* is a finite dimensional vector space.

$\epsilon_A \colon A^* \otimes A \to I$ be the unique linear maps satisfying:

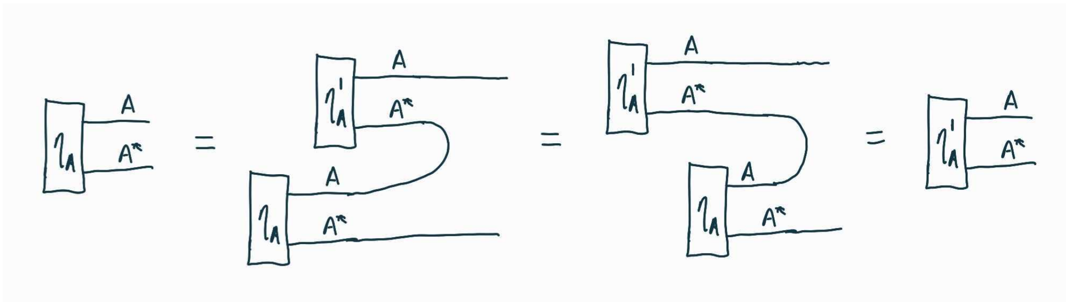
$$\eta_A(1) = \sum_{a \in \mathcal{B}_A} |a\rangle \otimes \langle a|$$
$$\epsilon_A(\langle a'| \otimes |a\rangle) = \langle a'|a\rangle$$

for all $a, a' \in A$. It is not hard to check — via explicit calculation — that η and ϵ satisfy the equations required for **FdHilb** to be a compact closed category under these definitions.

It may seem like the definition of η_A in the example of **FdHilb** is dependent on the choice of orthonormal basis $\mathcal{B}_A$; however, this is not the case. Proving so using linear algebra isn't hard, but this is an example where the graphical calculus of monoidal categories simplifies the proof greatly.

Proposition 4.4. *Let $(\mathbf{C}, \otimes, I)$ be a symmetric monoidal category and let $\eta_A \colon I \to A \otimes A^*$, $\eta'_A \colon I \to A \otimes A^*$ and $\epsilon_A \colon A^* \otimes A \to I$ be morphisms in it. Assume that η_A and ϵ_A satisfy (7); assume that η'_A and ϵ_A satisfy (7) as well. Then, $\eta_A = \eta'_A$.*

Proof. We prove this with string diagrams:



where (7) and the coherence theorem of monoidal categories have been used. □

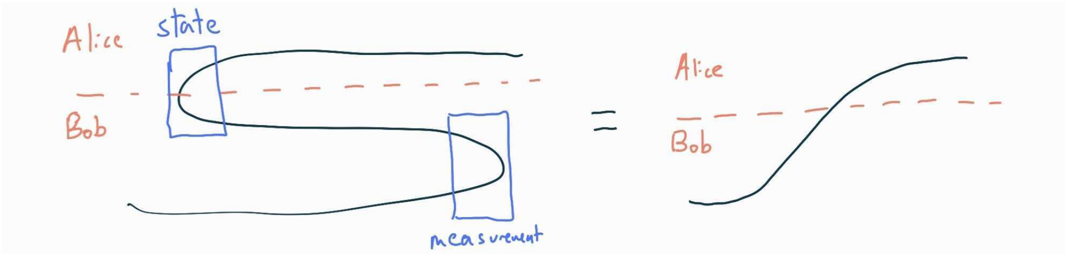
Since ϵ_A in Example 4.3 is independent from the choice of basis, it follows from the previous proposition that no matter which orthonormal basis is chosen when defining η_A, the result is the same morphism. However, this does not mean that given a symmetric monoidal category there is only one possible compact closed structure on it. The example below provides a different compact closed structure on **FdHilb**.

Example 4.5. For each finite dimensional Hilbert space $A \in$ **FdHilb** let $A^* = A$ and let $\mathcal{B}_A$ be an orthonormal basis. Let $\eta_A \colon I \to A \otimes A^*$ and $\epsilon_A \colon A^* \otimes A \to I$ be the following linear maps:

$$\eta_A(z) = z \sum_{a \in \mathcal{B}_A} |a\rangle \otimes |a\rangle$$
$$\epsilon_A = \sum_{a \in \mathcal{B}_A} \langle a| \otimes \langle a|$$

for all $a, a' \in A$. It is not hard to check that η and ϵ satisfy the equations required for **FdHilb** to be a compact closed category under these definitions.

Notice that $\eta_A(1)$ in the previous example corresponds (up to normalisation) to a maximally entangled state. Similarly, ϵ_A corresponds (up to normalisation) to a functional post-selecting output $\eta_A(1)$. Then, (7) describe the basic intuition of quantum teleportation:



Of course, this is not the full picture of the quantum teleportation protocol since the post-selection operation is non-deterministic. The ZX-calculus [1] can help with that, but this is not within the scope of this course.

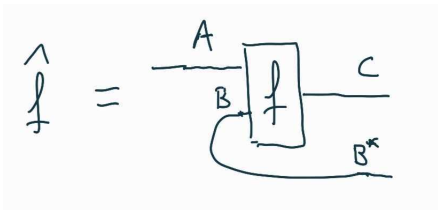
The following proposition introduces one of the fundamental properties of compact closed categories.

Proposition 4.6. *Let $(\mathbf{C}, \otimes, I)$ be a compact closed category and define $B \multimap C = C \otimes B^*$ and $\mathrm{ev}_{B,C} = \rho_C \circ (\mathrm{id}_C \otimes \epsilon_B) \circ \alpha_{C,B^*,B}$ for each $B, C \in \mathbf{C}$. For each morphism $f\colon A \otimes B \to C$ in $\mathbf{C}$ there is a unique morphism $\hat{f}\colon A \to (B \multimap C)$ such that the following diagram*

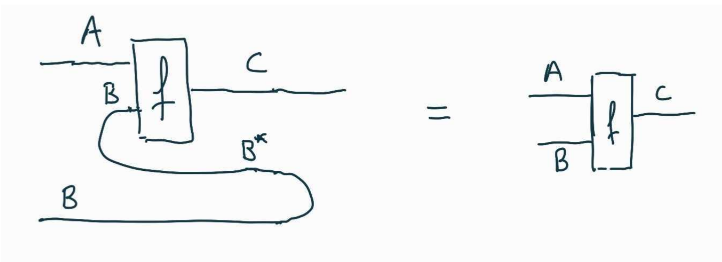
$$\begin{array}{ccc} (B \multimap C) \otimes B & \xrightarrow{\mathrm{ev}_{B,C}} & C \\ \uparrow{\scriptstyle \hat{f}\otimes \mathrm{id}_B} & \nearrow{\scriptstyle f} & \\ A \otimes B & & \end{array}$$

commutes in $\mathbf{C}$.

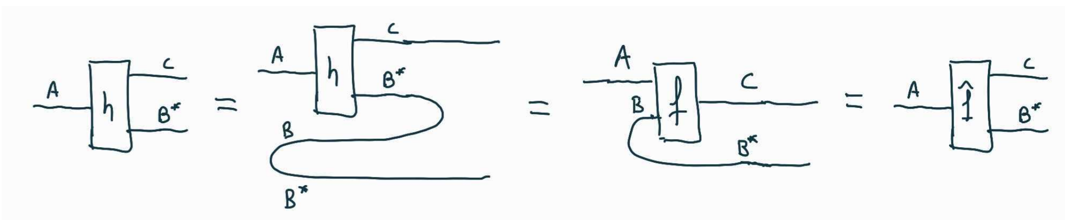
Proof. First, let's show that such a morphism $\hat{f}$ exists by defining



and checking that, indeed, the diagram in the claim commutes:



by direct application of (7). To prove uniqueness, let $h\colon A \to (B \multimap C)$ be an arbitrary morphism making the diagram commute. Then,

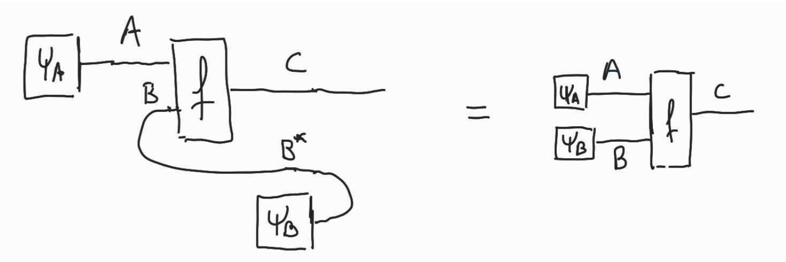


implying that $\hat{f}$ is unique. ∎

The proposition characterises $B \multimap C$ via a universal property. In particular, it establishes a bijection

$$\mathbf{C}(A \otimes B, C) \cong \mathbf{C}(A, B \multimap C). \tag{8}$$

So that each morphism $\hat{f}\colon A \to (B \multimap C)$ can be interpreted as mapping $a \mapsto f(a, -)$. This property is what 'closed' in 'compact closed category' refers to. There are many examples of categories that are closed but not *compact* closed; for instance, $(\mathbf{Set}, \times, \{\bullet\})$ is 'Cartesian closed' and such a structure is what makes 'currying' possible in programming languages. One of the multiple consequences of the previous proposition is that 'currying' is also available in quantum computing. In particular, for product states $|\psi_A\rangle \otimes |\psi_B\rangle$ it implies that we may provide the input on B at a later point in time (if we interpret time going from left to right in diagrams):



which, under the interpretation of Example 4.5, tells us that we require preparation of a maximally entangled state for η and post-selection for ϵ. Once again, since post-selection is non-deterministic, what we get is some kind of non-deterministic currying.[14]

Another consequence of **FdHilb** being compact closed is that the set of linear maps $\mathbf{FdHilb}(A, B)$ can be given the structure of a Hilbert space. Intuitively, $A \multimap B$ carries the same information as $\mathbf{FdHilb}(A, B)$ but, since $A \multimap B \in \mathbf{FdHilb}$, it follows that $\mathbf{FdHilb}(A, B)$ can be made into a Hilbert space. The formal proof is given below.

Proposition 4.7. *For each $A, B \in \mathbf{FdHilb}$, there is a Hilbert space of linear maps $A \to B$.*

Proof. Let's start from (8), relabelling some objects:

$$\mathbf{FdHilb}(\mathbb{C} \otimes A, B) \cong \mathbf{FdHilb}(\mathbb{C}, A \multimap B).$$

Recall that $A \cong \mathbb{C} \otimes A$ due to the left unitor being an isomorphism by the definition of monoidal categories. This implies that

$$\mathbf{FdHilb}(A, B) \cong \mathbf{FdHilb}(\mathbb{C} \otimes A, B)$$

since the function mapping each $f\colon A \to B$ to $f \circ \lambda_A$ is a bijection.[15] Moreover, notice that for all $X \in \mathbf{FdHilb}$ there is a bijection $\mathbf{FdHilb}(\mathbb{C}, X) \cong X$ since each linear map $f \in \mathbf{FdHilb}(\mathbb{C}, X)$ is uniquely characterised by the vector $f(1) \in X$; consequently:

$$\mathbf{FdHilb}(\mathbb{C}, A \multimap B) \cong A \multimap B.$$

[14] Notice that if f is Clifford we can make it deterministic by applying some classically-controlled Pauli gates at the end.

[15] It is injective since $f \circ \lambda_A = g \circ \lambda_A$ implies $f = f \circ \lambda_A \circ \lambda_A^{-1} = g \circ \lambda_A \circ \lambda_A^{-1} = g$. It is surjective since any $h\colon I \otimes A \to B$ has $h \circ \lambda_A^{-1}$ as its preimage.

Since $A \multimap B = B^* \otimes A$ by definition, we conclude that there is a bijection of sets $\mathbf{FdHilb}(A, B) \cong B \otimes A^*$. Moreover, since $B \otimes A^*$ is a Hilbert space — by virtue of being an object of **FdHilb** — its addition, scalar multiplication and inner product can be pushed through the bijection $\mathbf{FdHilb}(A, B) \cong B \otimes A^*$ to turn the set $\mathbf{FdHilb}(A, B)$ into a Hilbert space. □

Now that we have verified that $\mathbf{FdHilb}(A, B)$ is a Hilbert space, we may consider *superoperators*, *i.e.* transformations from linear maps to linear maps. Linear superoperators will correspond to morphisms in **FdHilb** of type $(A \multimap B) \to (C \multimap D)$. This puts us in the right direction towards the study of density matrices (which are elements in $A \multimap A$) and completely positive maps. However, we are lacking a key ingredient: a categorical formalisation of positivity. This will be achieved in the following subsection.

As a final note: the category $(\mathbf{Hilb}, \otimes, \mathbb{C})$ is *not* compact closed. The intuitive reason being that linear maps η_A and ϵ_A from Examples 4.3 and 4.5 cannot be bounded if A is infinite dimensional and, hence, they are not morphisms in **Hilb** — the morphisms in **Hilb** are *bounded* linear maps.

4.2 Dagger categories

Definition 4.8. Let $\mathbf{C}$ be a category equipped with an operation $\dagger\colon \mathbf{C}(A, B) \to \mathbf{C}(B, A)$ for each $A, B \in \mathbf{C}$ satisfying the following:

- $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$,
- $\mathrm{id}_A^\dagger = \mathrm{id}_A$,
- $(f^\dagger)^\dagger = f$

for every $f \in \mathbf{C}(A, B)$, $g \in \mathbf{C}(B, C)$ and $A, B, C \in \mathbf{C}$.

Example 4.9. **Rel** is a dagger category where for each morphism $\mathcal{R}\colon A \to B$ the dagger yields a relation $\mathcal{R}^\dagger$ characterised as follows:

$$b \mathcal{R}^\dagger a \iff a \mathcal{R} b$$

for all $a \in A$ and $b \in B$.

Example 4.10. Both **FdHilb** and **Hilb** are dagger categories where the dagger of a morphism $f\colon A \to B$ yields its adjoint linear map $f^\dagger\colon B \to A$.

As usual, a category may have multiple candidates of † that make it a dagger category. For instance, another way to make **FdHilb** a dagger category is to define † as transposition of the linear map's matrix. However, defining † as the adjoint is particularly useful. For instance, let $A \in \mathbf{FdHilb}$ and $v, u \in A$; the usual inner product of v and u is:

$$\langle v | u \rangle = v^\dagger \circ u$$

where v and u are seen as morphisms in $\mathbf{FdHilb}(\mathbb{C}, A)$. Moreover, we can identify positive linear maps via the following property.

Proposition 4.11. *A linear map $f \in \mathbf{FdHilb}(A, A)$ is* positive *if and only if there is some $B \in \mathbf{FdHilb}$ and some $g \in \mathbf{FdHilb}(A, B)$ such that*

$$f = g^\dagger \circ g.$$

Proof. The usual definition of positive linear map establishes that $f\colon A \to A$ is positive if and only if for all $a \in A$:

$$\langle a \,|\, f(a) \rangle \geq 0.$$

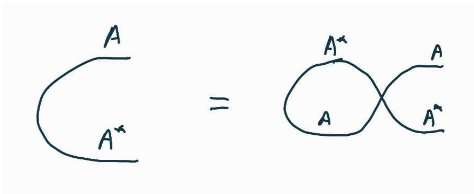
It is trivial to prove that this is satisfied if $f = g^\dagger \circ g$ since:

$$\langle a \,|\, f(a) \rangle = \langle a \,|\, g^\dagger \circ g(a) \rangle = \langle g(a) \,|\, g(a) \rangle \geq 0$$

where the properties of the adjoint $g^\dagger$ and the inner product's positive semi-definiteness have been used. To prove the other direction, assume $\langle a \,|\, f(a) \rangle \geq 0$ for all $a \in A$ and recall that this implies that f is self-adjoint.[16] Then, the spectral theorem establishes that f can be diagonalised as $f = u^\dagger d u$ where u is unitary and d is given by a diagonal matrix of real entries. Let $g = \sqrt{d} \circ u$, then $f = g^\dagger \circ g$. □

Finally, let's put the dagger structure and the compact closed structure together.

Definition 4.12. A category $(\mathbf{C}, \otimes, I)$ is *dagger compact closed* if it is both a dagger category and a compact closed category and satisfies that $\eta_A = \sigma_{A^*,A} \circ \epsilon_A^\dagger$; graphically:



.

Example 4.13. **Rel** is a dagger compact closed category, with its compact closed structure given in Example 4.2 and its dagger structure given in Example 4.9.

Example 4.14. **FdHilb** is a dagger compact closed category, with its compact closed structure given in Example 4.5 and its dagger structure given in Example 4.10. Notice that the other dagger and compact closed structures on **FdHilb** discussed in this section fail to make it a dagger compact closed category.

The exercises of this final section are based on [7] and prove that every completely postive superoperator $(A \multimap A) \to (B \multimap B)$ in **FdHilb** is uniquely characterised by a positive linear map. This is a fundamental part of the Choi-Jamiołkowski isomorphism between CPTP maps and density matrices; what remains is to study trace preservation and — although this can be neatly captured in the framework of compact closed categories; see [3] — it is beyond the scope of this course.

4.3 Exercises

A density matrix on a Hilbert space A is a linear map $A \to A$ that is positive and whose trace equals one. General quantum processes are linear maps that send density matrices to density matrices and, hence, are morphisms in **FdHilb** of type $(A \multimap A) \to (B \multimap B)$. In these exercises we will focus on the subproblem of studying superoperators that send positive linear maps to positive linear maps.

[16] The statement $\langle a \,|\, f(a) \rangle \geq 0$ tells us that $\langle a \,|\, f(a) \rangle$ is a *real* positive number. Then,

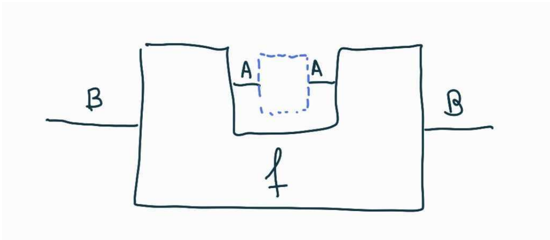
$$\langle a \,|\, f(a) \rangle = \overline{\langle a \,|\, f(a) \rangle} = \langle f(a) \,|\, a \rangle = \langle a \,|\, f^\dagger(a) \rangle$$

so it follows that $\langle a \,|\, f(a) \rangle - \langle a \,|\, f^\dagger(a) \rangle = 0$. By linearity, $\langle a \,|\, f(a) - f^\dagger(a) \rangle = 0$ for all $a \in A$ so it must be that $f(a) - f^\dagger(a) = 0$ and, hence, $f = f^\dagger$.

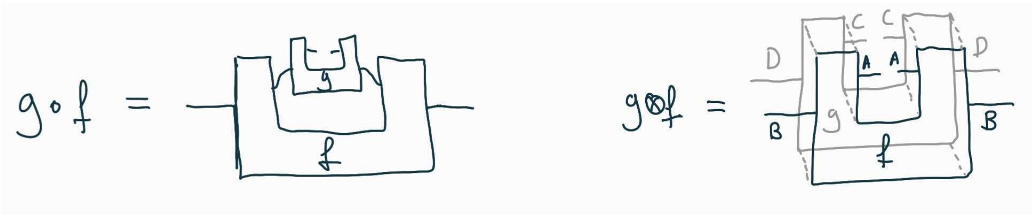
Definition 4.1. A *completely positive superoperator* is a morphism $f\colon (A \multimap A) \to (B \multimap B)$ in **FdHilb** such that for all $C \in$ **FdHilb** and all linear maps $\rho \in (C \otimes A) \multimap (C \otimes A)$ satisfies:

$$\rho \text{ is positive } \implies (\mathrm{id} \otimes f)(\rho) \text{ is positive.}$$

If boxes in **FdHilb** depict linear maps $A \to A$ it is reasonable to depict a superoperator $(A \multimap A) \to (B \multimap B)$ as a box with a hole where the input linear map $A \to A$ should go; in the literature, these are known as *combs*.

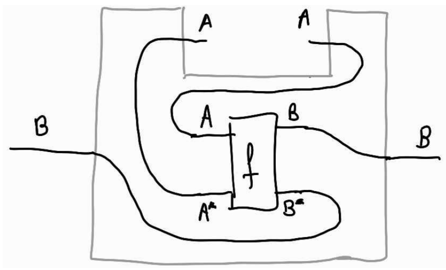


Composition of superoperators and tensor product can also be depicted using some imagination:



.

Moreover, we know from previous discussion in this section that a superoperator $(A \multimap A) \to (B \multimap B)$ is just a linear map from $A \otimes A^*$ to $B \otimes B^*$. However, in order to depict it in the form of a comb we need to make use of the isomorphism $\mathbf{FdHilb}(X, Y) \cong Y \otimes X^*$ to get the types right. Doing so leads to the following internal wiring of a comb:



.

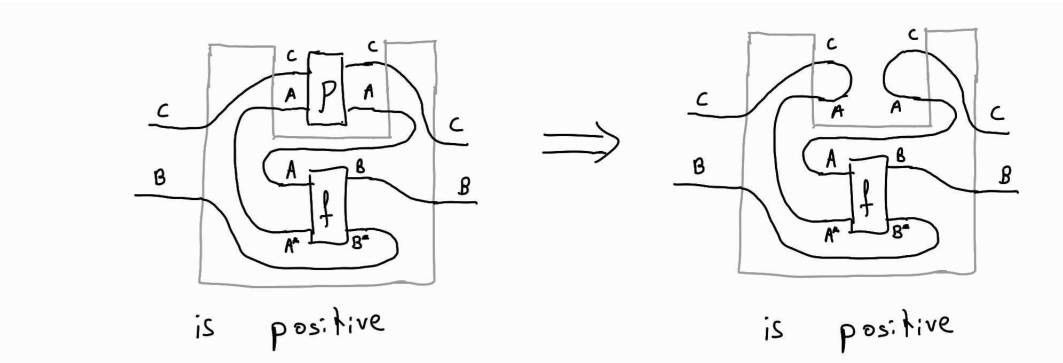
[!] **4.2.** (a) Draw the internal wiring of the comb that corresponds to the linear map $\mathrm{id}\colon A \otimes A^* \to A \otimes A^*$. Simplify it using (7).

(b) Draw the internal wiring of the comb of $\mathrm{id}_{C \multimap C} \otimes f$ where f is a superoperator of type $(A \multimap A) \to (B \multimap B)$. **Hint**: you should get something that looks like the comb of the next exercise.

We are now in a position to tackle the following exercise, which is a key component of the Choi-Jamiołkowski isomorphism theorem.

[!] **4.3.** Let $f\colon A \otimes A^* \to B \otimes B^*$ be a morphism in **FdHilb**.

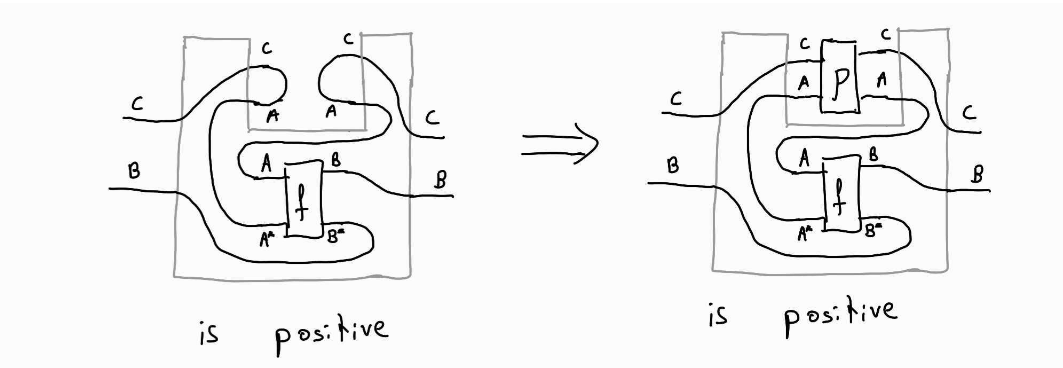
(a) Prove that for all $C \in \mathbf{FdHilb}$ and all positive linear maps $\rho\colon C \otimes A \to C \otimes A$:



.

Hint: Using Proposition 4.11, identify a positive linear map of type $C \otimes A \to C \otimes A$ within the right hand side of the implication. Once you do, the right hand side follows immediately from the left hand side.

(b) Prove that for all $C \in \mathbf{FdHilb}$ and all positive linear maps $\rho\colon C \otimes A \to C \otimes A$:



.

Hint: Simplify the left hand side using (7); then, apply Proposition 4.11 on the result and substitute in the right hand side. After some further diagram manipulation you should be able to use Proposition 4.11 to prove that the right hand side is positive.

(c) Conclude that every completely postive superoperator $(A \multimap A) \to (B \multimap B)$ is uniquely characterised by a positive linear map $A^* \otimes B \to A^* \otimes B$.

References

[1] Bob Coecke and Aleks Kissinger. *Picturing Quantum Processes: A First Course in Quantum Theory and Diagrammatic Reasoning*. Cambridge University Press, 2017.

[2] Chris Heunen and Robin Kaarsgaard. Quantum information effects. *Proc. ACM Program. Lang.*, 6(POPL), jan 2022.

[3] Chris Heunen and Jamie Vicary. *Categories for Quantum Theory: an introduction*. Oxford University Press, 2019.

[4] Mathieu Huot and Sam Staton. Universal properties in quantum theory. *arXiv preprint arXiv:1901.10117*, 2019.

[5] Tom Leinster. *Basic Category Theory*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2014.

[6] Emily Riehl. *Category theory in context*. Courier Dover Publications, 2017.

[7] Peter Selinger. Dagger compact closed categories and completely positive maps: (extended abstract). *Electronic Notes in Theoretical Computer Science*, 170:139–163, 2007. Proceedings of the 3rd International Workshop on Quantum Programming Languages (QPL 2005).